Final Exam Review Questions

These questions are intended as a review for the final exam. There will be just one "hard proof" on the final, and it will be selected from questions 1-3 below. There will also be one question on differential equations, and it will be similar to one of questions 48-50 below.

1. If $v_1, \dots, v_n$ is a basis for $\mathbf{R}^n$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ is a one-to-one linear transformation, prove that $T(v_1), \dots, T(v_n)$ is also a basis for $\mathbf{R}^n$.

2. Let $S = \{e_1, \dots, e_n\}$ be an orthonormal basis for a finite-dimensional Euclidean space V. Prove that for every $x \in V$, we have
$$x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

3. Let W be a subspace of a finite-dimensional Euclidean space V, and let $p(x)$ denote the orthogonal projection of an element $x \in V$ onto W. Prove that
$$\|x - p(x)\| \leq \|x - y\|$$
for all $y \in W$, with equality if and only if $y = p(x)$.

4. Define what it means for a set S of vectors in a linear space V to be *linearly independent.*

5. Define what it means for a linear space V to be *finite-dimensional.*

6. Define what it means for a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ to be *diagonalizable.*

7. Define what it means for a sequence $\{a_n\}$ of real numbers to *converge.*

8. Determine all solutions to the system
$$\begin{aligned} 5x + 2y - 6z + 2u &= -1 \\ x - y + z - u &= -2 \end{aligned}$$

9. Find an equation of the form $y = ax^2 + bx + c$ for the unique parabola passing through the points $(0, 0), (1, 1), (4, 8)$.

10. Find a scalar parametric equation for the plane through the three points $(2, 3, 1), (-2, -1, -3)$, and $(4, 3, -1)$.

11. Show that for any two vectors $A, B \in \mathbf{R}^n$ we have

$$\|A + B\|^2 - \|A - B\|^2 = 4A \cdot B.$$

12. Find the reduced row echelon form and the rank of the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 \\ 1 & -4 & 2 & -2 \end{pmatrix}$$

13. Find all real t for which the two vectors $(t + 1, t)$ and $(t, t + 1)$ are linearly dependent.

14. Is $S = \{(x, y, z) \in \mathbf{R}^3 \ : \ x = y = z - 1\}$ a subspace of $\mathbf{R}^3$? Why or why not?

15. Let P_n denote the linear space of all real polynomials of degree at most n. Is $S = \{f \in P_n \ : \ f''(0) = 3f(0)\}$ a subspace of P_n?

16. Find a basis for the subspace W of $\mathbf{R}^4$ spanned by the vectors

$$(1, 1, 1, 1),\ (2, 2, 2, 2)\ (1, -1, 1, -1)\ (2, 0, 2, 0),$$

and determine the dimension of W.

17. Let P_2 be the linear space of all polynomials of degree at most 2. Find the coordinates of $t^2 - 2t + 3$ relative to the ordered basis $(1, 3t - 1, t^2 - t)$ for P_2.

18. Find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by $(1, 1, 0, 0), (0, 1, 1, 1)$, and $(1, 0, 0, 1)$.

19. Let $W \subseteq \mathbf{R}^4$ be the linear span of $(1, 0, 0, 1)$ and $(2, 3, 1, 0)$. Find a basis for the orthogonal complement of W (with respect to the standard inner product).

20. In the real linear space $C(0,1)$ with inner product given by $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$, find the linear polynomial g closest to the function $f(x) = e^x$.

21. Let

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 0 & 1 \end{pmatrix},\ b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find the best approximate (least squares) solution to the overdetermined system of equations $Ax = b$.

22. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation for which $T(1,0) = (1,0,1)$ and $T(0,1) = (-1,0,1)$. Determine the rank and nullity of T.

23. Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

24. Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

25. Find all (real or complex) eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

26. Is the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{pmatrix}$$

diagonalizable? Justify your answer.

27. Find a nonsingular matrix C and a diagonal matrix D for which $D = C^{-1}AC$, where

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

28. If T is the linear transformation whose matrix with respect to the standard ordered basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 1 & 7 \\ 2 & 3 \end{pmatrix},$$

find the matrix of T with respect to the ordered basis $b_1 = (1, -1), b_2 = (2, 1)$ of $\mathbf{R}^2$.

29. Let T be the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by reflection around the line $y = 2x$.

    (a) Find the matrix of T with respect to the ordered basis $b_1 = (1, 2), b_2 = (-2, 1)$ of $\mathbf{R}^2$.

    (b) Use part (a) and the change-of-basis formula to find the matrix of T with respect to the standard ordered basis e_1, e_2 of $\mathbf{R}^2$.

30. Compute a closed-form formula for A^k, where k is a positive integer and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

31. Show that $\frac{3^n}{n!} \to 0$.

32. Evaluate $\lim_{n\to\infty}(3 + \frac{4}{n})^{2n}$.

33. Evaluate $\lim_{x\to\infty} \frac{1+x-e^x}{x(e^x-1)}$.

34. Evaluate $\lim_{n\to\infty}(\cos \frac{1}{n})^n$.

35. Evaluate $\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)}$.

36. Does the series $\sum_{k=1}^{\infty} \frac{2k+1}{\sqrt{k^5+1}}$ converge or diverge? Justify your answer.

37. Does the series $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ converge or diverge? Justify your answer.

38. Does the series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k(k+1)}}$ converge absolutely, converge conditionally, or diverge? Justify your answer.

39. Give an upper bound for the error when the partial sum $\sum_{k=1}^{24} \frac{1}{\sqrt{k+1}}$ is used to estimate the sum of the infinite series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$.

40. Expand $g(x) = x^3 \sin(2x^2)$ as a power series in x.

41. Find the Taylor series for e^{-x} centered around $x = 1$, and prove that this series converges to e^{-x} for all real x.

42. Use the Lagrange form of the remainder for Taylor series to estimate $\sqrt{83}$ to within 3 decimal places.

43. Find the radius of convergence and interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{2^k}{k^2} x^k$.

44. Find the interval of convergence for the power series

$$\frac{x-1}{2} - \frac{2(x-1)^2}{4} + \frac{3(x-1)^3}{8} - \frac{4(x-1)^4}{16} + \cdots$$

45. Find the sum of the infinite series $\sum_{k=1}^{\infty} \frac{k}{5^k}$.

46. Find a simple closed-form expression for the function $f(x)$ given by the infinite series expansion $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+2}}{k!(k+2)}$.

47. Use power series to evaluate $\int_0^1 \sin(x^2)dx$ to within 2 decimal places.

48. Find a system of 3 linear first-order ODE's which is equivalent to the third-order ODE $y''' - 2y' + 3y = 0$.

49. Calculate e^A when

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

[**Hint:** What is A^3?]

50. Let

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}.$$

Compute e^{tA}, and use this to solve the system of differential equations $Y'(t) = AY(t)$, where

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

is subject to the initial conditions $y_1(0) = 1, y_2(0) = 1$.