HW assignment #2 (Due Thursday, September 4)

Read Sections IV.1 and IV.2 in the course textbook. Then do the following exercises:

1. (Knapp §IV.12 #1) Let G be a group in which all elements other than the identity have order 2. Prove that G is abelian.
2. (Knapp §IV.12 #2) The dihedral group D_4 of order 8 can be viewed as a subgroup of the symmetric group S_4. Find 8 explicit permutations in S_4 forming a subgroup isomorphic to D_4.
3. (Knapp §IV.12 #3) Suppose G is a finite group, H is a subgroup, and $a \in G$ is an element with $a^\ell \in H$ for some integer ℓ with $\mathrm{GCD}(\ell, |G|) = 1$. Prove that $a \in H$.
4. (Knapp §IV.12 #5) Prove that if G is an abelian group and n is an integer, then $a \mapsto a^n$ is a homomorphisms from G to itself. Give an example of a nonabelian group for which $a \mapsto a^2$ is not a homomorphism.
5. If G is a group and $a, b \in G$, prove that ab and ba have the same order.
6. If $n \geq 3$ is odd, show that the identity is the only element of the dihedral group D_n of order $2n$ which commutes with all elements of the group, i.e., that the *center* of D_n is trivial.
7. Let G be an abelian group.
   (a) Prove that the set of elements of finite order in G is a subgroup (called the *torsion subgroup* of G).
   (b) Give an explicit example where this set is not a subgroup when G is non-abelian.
   (c) Prove that $\mathbf{Q}/\mathbf{Z}$ is the torsion subgroup of $\mathbf{R}/\mathbf{Z}$, and that $\mathbf{Q}/\mathbf{Z}$ contains elements of arbitrarily large order.

8. Assume both H and K are normal subgroups of G with $H \cap K = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$.

9. Let H be a subgroup of G. Prove that the map $x \mapsto x^{-1}$ sends each left coset of H in G onto a right coset of H and gives a bijection between the set of left cosets and the set of right cosets of H in G. In particular, deduce that the number of left cosets of H in G equals the number of right cosets.

10. Let G be a group. Prove that the subgroup N generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in G$ is a normal subgroup of G, and that G/N is abelian.