HW ASSIGNMENT #3 (DUE THURSDAY, SEPTEMBER 11)

Read Section IV.6 in the course textbook. Then do the following exercises:

1. (Knapp §IV.12 #25) Let G be a group, and let H, K be subgroups of G.
   (a) For $x, y \in G$, prove that $xH \cap yK$ is either empty, or is a coset of $H \cap K$ in G.
   (b) Deduce from (a) that if H and K both have finite index in G, then so does $H \cap K$.
2. (Knapp §IV.12 #28) Let G be a group, and let H be a subgroup of G. Prove or disprove that the normalizer $N_G(H)$ of H in G is a normal subgroup of G.
3. Prove that the alternating group A_4 of order 12 has no subgroup of order 6.
4. Prove that any subgroup or quotient group of a cyclic group is again cyclic. [You may rewrite the proof of Proposition 4.4 in your own words for the subgroup part of this question.]
5. The *center* $Z(G)$ of a group G is the set $\{g \in G \mid gh = hg \text{ for all } h \in G\}$.
   (a) Show that $Z(G)$ is a normal subgroup of G.
   (b) If G is non-abelian, prove that the group $G/Z(G)$ is not cyclic.
6. Let G be a cyclic group.
   (a) If G is infinite, prove that G has exactly one subgroup of index n for each positive integer n, and no other subgroups except for the trivial group.

(b) If G is finite of order n, prove that G has exactly one subgroup of order d for each positive integer d dividing n, and no other subgroups.

7. Let G be a group, and let H be a subgroup of G.

   (a) Verify that G acts by left multiplication on the left coset space G/H.

   (b) Show that this action is transitive, and compute the isotropy group of the trivial coset H.

   (c) Prove that the kernel K of this action is contained in H, and more generally that it is the largest normal subgroup of G contained in H. [**Hint:** Show that $K = \cap_{g \in G}\, gHg^{-1}$.]

   (d) If G is a group of order 140 and H is a subgroup of G of order 35, prove that H is normal in G. [**Hint:** Use part (c) and consider the permutation representation $\phi : G/K \to S_4$ associated to the action of G on G/H.]