

HW ASSIGNMENT #5 (DUE THURSDAY, OCTOBER 2)

Read Sections IV.7 and IV.10 in the course textbook. Then do the following exercises:

1. (Knapp §IV.12 #29) How many elements of order 7 are there in a simple group of order 168?
2. (Knapp §IV.12 #32) Prove that a group of order 24 cannot be simple.
3. How many Sylow 2-subgroups does S_5 have?
4. (a) Show that a p -Sylow subgroup of the dihedral group D_n of order $2n$ is cyclic and normal for every odd prime p .
(b) If m is the largest odd divisor of n , show that D_n has exactly m 2-Sylow subgroups.
5. Let p be a prime, and let \mathbf{F}_p be the finite field with p elements. Show that $\text{GL}_2(\mathbf{F}_p)$ has exactly $p+1$ p -Sylow subgroups. [**Hint:** Look at the subgroup of upper triangular matrices.]
6. Let G be a group of order pqr , with $p < q < r$ distinct primes.
 - (a) Show that at least one of the Sylow subgroups of G is normal, and in particular that G is not simple.
 - (b) Using part (a), show that in fact the Sylow r -subgroup of G is normal.
7. Classify (up to isomorphism) all groups of order $2p$, where p is a prime number.
8. Let H be an abelian group, let K be a cyclic group of order 2 with generator s , and let K act on H by inversion via the rule $s \cdot h = h^{-1}$. Let G be the semidirect product of K and H with respect to this action. Show that every element of $G - H$ has order 2, and that G is abelian iff every non-identity element of H has order 2.

9. (Knapp §IV.12 #35-36) Let H, K be groups, and let $\varphi_1 : K \rightarrow \text{Aut}(H)$ and $\varphi_2 : K \rightarrow \text{Aut}(H)$ be homomorphisms.
- If $\varphi_2 = \varphi_1 \circ \varphi$ for some automorphism φ of K , prove that $K \times_{\varphi_1} H \cong K \times_{\varphi_2} H$. [**Hint:** Show that the map $\psi : K \times_{\varphi_2} H \rightarrow K \times_{\varphi_1} H$ defined by $\psi(k, h) = (\varphi(k), h)$ is an isomorphism.]
 - If $\varphi_2 = \varphi \circ \varphi_1$ for some inner automorphism φ of $\text{Aut}(H)$, prove that $K \times_{\varphi_1} H \cong K \times_{\varphi_2} H$. [**Hint:** Show that the map $\psi : K \times_{\varphi_1} H \rightarrow K \times_{\varphi_2} H$ defined by $\psi(k, h) = (k, \alpha(h))$ is an isomorphism, where $\varphi(\beta) = \alpha \circ \beta \circ \alpha^{-1}$ for $\beta \in \text{Aut}(H)$.]
10. (Knapp §IV.12 #37-38) Suppose p and q are primes.
- If $\mathbf{Z}/p\mathbf{Z}$ acts nontrivially on $\mathbf{Z}/q\mathbf{Z}$ by automorphisms, prove that $p \mid q - 1$.
 - If $p \mid q - 1$, prove that there is only one nonabelian semidirect product $\mathbf{Z}/p\mathbf{Z} \times_{\tau} \mathbf{Z}/q\mathbf{Z}$ up to isomorphism.