Math 6121 Fall 2008

HW ASSIGNMENT #6 (DUE THURSDAY, OCTOBER 16)

Read Sections IV.8 and IV.9 in the course textbook. Then do the following exercises:

1. For any field F, prove that the group of upper triangular matrices in $\mathrm{GL}_n(F)$ is a semidirect product $D \times_\tau U$, where D is the subgroup of diagonal matrices and U is the subgroup of upper triangular matrices with 1's down the diagonal. (If you are not comfortable with fields, you may assume that $F = \mathbf{R}$.)

2. (a) If G is a finite group such that every Sylow subgroup of G is normal and abelian, show that G is abelian.

   (b) Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal, then G is abelian.

3. A group G is called *solvable* iff all of its composition factors are abelian. Prove that if H is a normal subgroup of G, then G is solvable iff H and G/H are both solvable.

4. Show that for each $n \geq 3$, the $n-2$ 3-cycles $(123), (124), \ldots, (12n)$ generate A_n.

5. Show for each $n \geq 1$ that there is an injective homomorphism from S_n into A_{n+2}. Conclude that every finite group is isomorphic to a subgroup of A_n for some n.

6. (a) Find all normal subgroups of S_n for $n \geq 5$.

   (b) Prove that A_n is the only proper subgroup of index less than n in S_n for $n \geq 5$.

7. Write down a complete list of all abelian groups of order 270.

8. Find the number of subgroups of order 2 in $\mathbf{Z}/60\mathbf{Z} \times \mathbf{Z}/45\mathbf{Z} \times \mathbf{Z}/12\mathbf{Z} \times \mathbf{Z}/36\mathbf{Z}$.

9. (Knapp §IV.12 #21) Let G be the quotient of the free abelian group with $\mathbf{Z}$-basis x_1, x_2, x_3 by the subgroup $H = \langle 3x_1 + 2x_2 + 5x_3, x_2 + 3x_3, x_2 + 5x_3 \rangle$. Express G/H as a direct sum of cyclic groups.

10. For any group G, define its *dual group* $\hat{G}$ to be the set of all homomorphisms from G into $\mathbf{C}^*$, together with the binary operation of pointwise multiplication of functions.

    (a) Show that this binary operation makes $\hat{G}$ into an abelian group.

    (b) If G is a finite abelian group, prove that $\hat{G} \cong G$. [**Hint:** If $G \cong \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ with x_i of order n_i, show that $\hat{G} \cong \langle \chi_1 \rangle \times \cdots \times \langle \chi_r \rangle$, where χ_i sends x_i to $e^{2\pi i n_i}$ and x_j to 1 for $j \neq i$.] Thus every finite abelian group is self-dual. Note, however, that there is no *natural* isomorphism between G and its dual (the isomorphism depends on a choice of a set of generators for G).