HW ASSIGNMENT #7 (DUE THURSDAY, OCTOBER 30)

Read Sections IV.4 and IV.5 in the course textbook. Then do the following exercises:

1. (a) Find the group of units in the ring $\mathbf{H}_{\mathbf{R}}$ of (real) Hamilton quaternions. [**Hint:** Show that $\alpha = a + bi + cj + dk$ is a unit iff $N(\alpha) = a^2 + b^2 + c^2 + d^2$ is nonzero.]
   (b) The *center* of a ring R is the set of all elements which commute with every element of R. Describe the center of $\mathbf{H}_{\mathbf{R}}$.
2. (a) Find all ring homomorphisms from $\mathbf{Z}$ to $\mathbf{Z}/30\mathbf{Z}$. In each case, describe the kernel and image of the homomorphism.
   (b) Find all ring homomorphisms from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}$. In each case, describe the kernel and image of the homomorphism.
3. Let R and S be nonzero rings with identity. If S is an integral domain, prove that every nonzero ring homomorphism $\varphi : R \to S$ sends the identity of R to the identity of S. Find an example where the conclusion can fail if S is not an integral domain.
4. Let I, J be ideals in a ring R.
   (a) Prove that $I + J = \{x + y \ : \ x \in I, \, y \in J\}$ is the smallest ideal of R containing both I and J.
   (b) Prove that

   $$IJ = \{x_1 y_1 + \cdots + x_k y_k \ : \ x_1, \ldots, x_k \in I, \, y_1, \ldots, y_k \in J\}$$

   is an ideal contained in $I \cap J$. Give an example where $IJ \neq I \cap J$.
   (c) If R is a commutative ring with identity and $I + J = R$, prove that $IJ = I \cap J$.
5. An element x of a commutative ring R is called *nilpotent* if $x^m = 0$ for some positive integer m.

(a) If $x \in R$ is nilpotent and $y \in R$ is a unit, prove that $x + y$ is a unit.

(b) Prove that the set of nilpotent elements in R forms an ideal $\mathfrak{N}(R)$ (called the *nilradical* of R). Give an example where the conclusion can fail if R is not commutative.

(c) Prove that $R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.

6. Let F be a field, and let R be the ring $M_{n\times n}(F)$ of $n \times n$ matrices over F.

   (a) Prove that R is *simple*, i.e., that its only two-sided ideals are (0) and R.

   (b) If S is a ring and $\varphi : R \to S$ is a nonzero homomorphism, prove that φ is injective.

7. Let p be a prime number and R a ring with identity containing p^2 elements. Prove that R is commutative.

8. Let R be a subring of a commutative ring S, and suppose $|S/R| = n$ is finite. If m is an integer relatively prime to n, prove that R/mR and S/mS are isomorphic rings.