Math 6121 Fall 2008

HW assignment #9 (Due Thursday, November 20)

Read Sections VIII.4 and VIII.5 in the course textbook. Then do the following exercises:

1. Let R be a commutative ring with identity. We say that R satisfies the *ascending chain condition* (ACC) if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ of ideals in R eventually stabilizes.

   (a) Prove that R satisfies the ACC iff every ideal of R is finitely generated. (A ring satisfying either of these conditions is called *Noetherian.*)

   (b) Prove that a polynomial ring in infinitely many variables over a field F is not Noetherian.

2. Let n be a positive integer, and let f be a polynomial in $\mathbf{R}[X]$ of degree n. Prove that there are real numbers $a_0, a_1, \ldots, a_n$, not all equal to zero, such that the polynomial
$$\sum_{i=0}^{n} a_i X^{2^i}$$
is divisible by f. [Use the division algorithm and linear algebra.]

3. Let F be a field. Prove that $F[x]$ contains infinitely many irreducible (= prime) elements.

4. (Knapp §VIII.12 # 10) Determine the maximal ideals of each of the following rings:

   (a) $\mathbf{R} \times \mathbf{R}$

   (b) $\mathbf{R}[X]/(X^2)$

   (c) $\mathbf{R}[X]/(X^2 - 3X + 2)$

   (d) $\mathbf{R}[X]/(X^2 + X + 1)$

5. (a) Determine all ideals in the ring $\mathbf{Z}[X]/(2, X^3 + 1)$.

(b) Let I be the ideal $(5, X^3 + X + 1)$ of $\mathbf{Z}[X]$. Is I a prime ideal?

6. (a) Prove that the polynomial $X^4 + X + 1$ is irreducible over $\mathbf{Q}$.

   (b) Prove that the polynomial $X^4 + X^3 + X^2 + 6X + 1$ is irreducible over $\mathbf{Q}$.

   (c) Prove that the polynomial $X^3 + nX + 2$ is irreducible over $\mathbf{Q}$ for all integers $n \neq 1, -3, -5$.

7. Exhibit a finite field of each of the following orders: (a) 8, (b) 9, (c) 49, (d) 81. [Your answer should be of the form $\mathbf{F}_p[X]/(f(x))$ for some prime p and some polynomial $f(x)$.] Write out explicit addition and multiplication tables for the fields in parts (a) and (b).

8. Prove that the quotient ring $\mathbf{Z}[i]/I$ is finite for any nonzero ideal I of the ring $\mathbf{Z}[i]$ of Gaussian integers. [**Hint:** If $I = (\alpha)$, use the division algorithm to show that every coset of I is represented by an element of norm less than $N(\alpha)$.]

9. Let p be a prime number. Show that the element $p \in \mathbf{Z}[i]$ is irreducible iff $p \equiv 3 \pmod 4$. [**Hint:** If $p \equiv 1 \pmod 4$, show that -1 is a square in $(\mathbf{Z}/p\mathbf{Z})^*$, and therefore that p divides $(n+i)(n-i)$ for some integer n. If $p \mid (n+i)$, show that $p \mid (n-i)$, and vice-versa; conclude from this that if p is irreducible then $p \mid 2n$, and deduce a contradiction from this.]

10. Let $R = \mathbf{Z}[\sqrt{-n}]$ with n a squarefree integer greater than 3.

    (a) Prove that $2, \sqrt{-n}$, and $1 + \sqrt{-n}$ are all irreducible in R.

    (b) Prove that R is not a UFD. [**Hint:** Show that either $\sqrt{-n}$ or $1 + \sqrt{-n}$ is not prime.]

    (c) Exhibit a non-principal ideal in R.

11. Find all integer solutions to the Diophantine equation $x^3 - y^2 = 2$. [**Hint:** Rewrite the equation as $y^2 + 2 = x^3$ and factor the left-hand side in the ring $\mathbf{Z}[\sqrt{-2}]$.]