Polynomial Problems (due Tuesday, October 16)

1. Determine all polynomials $P(x)$ with complex coefficients such that $P(0) = 0$ and $P(x^2 + 1) = (P(x))^2 + 1$.

2. Suppose the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has integer coefficients $a, b, c, d$ with $ad$ odd and $bc$ even. Prove that $f(x)$ has at least one irrational root.

3. Let $P(x)$ be a polynomial with real coefficients, and let $Q(x) = x^3P(x) + x^2 + x + 1$. Prove that $Q(x)$ cannot have all real roots.

4. Let $n$ be a positive integer, and let $f(x)$ be a degree $n$ polynomial with real coefficients. Prove that there are real numbers $a_0, a_1, \ldots, a_n$, not all zero, such that the polynomial

$$g(x) = \sum_{k=0}^{n} a_k x^{2^k}$$

is divisible by $f(x)$.

5. Do there exist polynomials $a(x), b(x), c(y), d(y)$ with complex coefficients such that

$$1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)?$$

6. Prove that for any polynomial $p(x)$ of degree at least 2 with integer coefficients, there exists polynomial $q(x)$ with integer coefficients for which the polynomial $p(q(x))$ can be factored into a product of non-constant polynomials with integer coefficients.