Infinite Series Problems (due Tuesday, October 23)

1. Evaluate the infinite sum
\[ \sum_{n=1}^{\infty} \frac{n^2}{2^n}. \]

2. Suppose that a sequence \(a_1, a_2, a_3, \ldots\) of positive real numbers satisfies
\[ a_n \leq a_{2n} + a_{2n+1} \] for all \(n \geq 1\). Prove that the series \(\sum_{n=1}^{\infty} a_n\) diverges.

3. Let \(N\) be the set of all positive integers which do not contain the digit 9 in their decimal representation. Prove that
\[ \sum_{a \in N} \frac{1}{a} < 80. \]

4. Let \(P\) be the set of all positive integers of the form \(a^b\) with \(a, b \geq 2\) integers. Prove that
\[ \sum_{q \in P} \frac{1}{q-1} = 1. \]

5. Prove that if \(\sum_{n=1}^{\infty} a_n\) is a convergent series of positive real numbers, then so is \(\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}\).

6. Show that
\[ \int_{0}^{1} \frac{1}{x^x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n} \]
(where by convention we set \(0^0 = 1\)).