Infinite Series Problems (due Tuesday, September 22)

1. a. Find a sequence a_n of positive real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

both converge.

b. Prove that there is no sequence a_n of positive real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

both converge.

2. a. Show that there is a sequence $\{\epsilon_n\}$ of plus and minus 1's such that

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{n} = \pi.$$

b. Show that for every sequence $\{\epsilon_n\}$ of plus and minus 1's,

$$\sum_{n=0}^{\infty} \frac{\varepsilon_n}{n!}$$

is irrational.

3. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \left(1 - \sqrt[n]{x}\right) = (1 - x) + (1 - \sqrt{x}) + (1 - \sqrt[3]{x}) + \cdots$$

converge?

4. Sum the infinite series

$$\sum_{n=1}^{\infty} \sin \frac{2\alpha}{3^n} \sin \frac{\alpha}{3^n}.$$

5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = x f_n'(x)$ for $n \geq 0$. Prove that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

6. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

7. Evaluate

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \right)^n.$$