Extension Theorems for Homomorphisms

In this note, we prove some extension theorems for homomorphisms from rings to algebraically closed fields. The prototype is the following result:

**Theorem 1** (Extension theorem for algebraic extensions). *If* \( L/K \) *is an algebraic extension of fields, then any embedding* \( \sigma \) *of* \( K \) *into an algebraically closed field* \( k \) *can be extended to an embedding* \( \tilde{\sigma} : L \to k \).

This was proved in class using Zorn’s Lemma. Note that some hypothesis akin to “\( k \) is algebraically closed” is needed for such a statement, since e.g. if \( L = K(\alpha) \) and \( f_\alpha(x) \) is the minimal polynomial of \( \alpha \) over \( K \), then \( \tilde{\sigma} \) must map \( \alpha \) to a root of \( f_\alpha(x) \) in \( k \). Since \( f_\alpha \) can be any monic polynomial in \( K[x] \), we need \( K \) to be algebraically closed in \( k \). A simple way to guarantee this is to require that \( k \) itself be algebraically closed.

We now prove an important generalization of Theorem 1 from algebraic extensions of fields to integral extensions of rings:

**Theorem 2** (Extension theorem for integral extensions). *Let* \( A \subseteq B \) *be rings with* \( B \) *integral over* \( A \). *Then every homomorphism* \( \phi \) *of* \( A \) *to an algebraically closed field* \( k \) *can be extended to a homomorphism* \( \tilde{\phi} : B \to k \).

**Proof.** Let \( p = \ker(\phi) \), and let \( S = A \setminus p \). Since \( \phi \) factors through the canonical homomorphism \( A \to S^{-1}A \) (by the universal property of localization), \( S^{-1}B \) is integral over \( S^{-1}A \) (by problem 3(a) on page 2 of Homework 1), and the diagram

\[
\begin{array}{ccc}
B & \longrightarrow & S^{-1}B \\
\uparrow & & \uparrow \\
A & \longrightarrow & S^{-1}A
\end{array}
\]

is commutative (again by the universal property of localization), we are reduced to the case where \( A \) is a local ring and \( \ker(\phi) \) is the unique maximal ideal \( m \) of \( A \) (by replacing \( A \) with \( A_p = S^{-1}A \) and \( B \) with \( S^{-1}B \)). By the Going-Up Theorem (Theorem 3 below), there is a maximal ideal \( M \) of \( B \) lying over \( m \) (i.e., such that \( M \cap A = m \)). Since \( B/A \) is integral, \( B/M \) is an algebraic extension of \( A/m \). By Theorem 1, we can extend \( \phi : A/m \to k \) to a homomorphism \( \tilde{\phi} : B/M \to k \).
Composing with the natural projection maps $A \to A/m$ and $B \to B/M$ gives the desired extension of $\phi$:

$$
\begin{array}{ccc}
B & \longrightarrow & B/M \\
\uparrow & & \uparrow \phi \\
A & \longrightarrow & A/m
\end{array} =
\begin{array}{ccc}
k & & k \\
\uparrow & & \uparrow \\
k & & k
\end{array}
$$

In the proof, we used:

**Theorem 3** (Going-Up Theorem). If $A \subseteq B$ are rings with $B$ integral over $A$, and $p$ is a prime ideal of $A$, then there exists a prime ideal $q$ lying over $p$, i.e., such that $q \cap A = p$. Furthermore, $q$ is maximal iff $p$ is maximal.

For this, we need a lemma:

**Lemma 1.** If $A \subseteq B$ are rings with $B$ integral over $A$, $q$ is a prime ideal of $B$, and $p = q \cap A$, then $q$ is maximal iff $p$ is maximal.

*Proof.* Since $B/q$ is integral over $A/p$, and both are integral domains, we are reduced to proving the following statement: If $A \subseteq B$ are integral domains with $B$ integral over $A$, then $A$ is a field iff $B$ is a field.

To see this, suppose first that $A$ is a field. If $y \in B$ is nonzero, let $y^n + a_{n-1}y^{n-1} + \cdots + a_0$ be an integral dependence of minimal degree, with $a_i \in A$. As $B$ is a domain, $a_0 \neq 0$, so $y^{-1} = -a_0^{-1}(y^{n-1} + \cdots + a_1) \in B$. Thus $B$ is a field.

Conversely, if $B$ is a field and $x \in A$ is nonzero, then $x^{-1} \in B$, hence is integral over $A$, so we have

$$x^{-m} + a'_{m-1}x^{-m+1} + \cdots + a'_0 = 0$$

with $a'_i \in A$. Thus $x^{-1} = -(a'_{m-1} + a'_{m-2}x + \cdots + a'_0x^{m-1}) \in A$, so $A$ is a field.

*Proof of Theorem 3.* Let $S = A \setminus p$. As before, we have a commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\beta} & S^{-1}B \\
\uparrow & & \uparrow \\
A & \xrightarrow{\alpha} & S^{-1}A
\end{array}
$$

with $S^{-1}B$ is integral over $S^{-1}A = A_p$. Let $M$ be any maximal ideal of $S^{-1}B$. Then $\mathfrak{m} = M \cap A_p$ is a maximal ideal in the local ring $A_p$ by
Lemma 1, hence equals $pA_p$. Under the correspondence between prime ideals of $S^{-1}B$ and prime ideals of $B$ disjoint from $S$, if $q = \beta^{-1}(M)$ is the prime ideal of $B$ corresponding to $M$, one sees from the commutativity of the above diagram that $q \cap A = \alpha^{-1}(m) = \alpha^{-1}(pA_p) = p$. □

One can use the Extension Theorem for Integral Extensions to give an insightful proof of the Algebraic Nullstellensatz:

**Theorem 4** (Algebraic Nullstellensatz). If $L/K$ is a field extension and $L$ is finitely generated as a $K$-algebra, then $L$ is a finite extension of $K$.

**Proof.** Let $x_1, \ldots, x_n$ be generators for $L$ as a $K$-algebra. If $L/K$ is algebraic, then (since $L/K$ is finitely generated) it must be a finite extension, and we’re done. So let’s assume, for the sake of contradiction, that $L/K$ is transcendental.

Let $t_1, \ldots, t_r \in L$ be a transcendence basis for $L/K$, i.e., $K[t_1, \ldots, t_r]$ is isomorphic to a polynomial ring in $r$ variables, and $x_1, \ldots, x_n$ are algebraic over $K(t_1, \ldots, t_r)$. (Convince yourself that such a basis always exists.) Let $p_i(X)$ be the minimal polynomial of $x_i$ over $K(t_1, \ldots, t_r)$. Multiplying by a suitable nonzero element $h$ of $K[t_1, \ldots, t_r]$ to clear denominators, we obtain a polynomial $q_i(X) = h \cdot p_i(X) \in K[t_1, \ldots, t_r][X]$ with $q_i(x_i) = 0$. Let $f = f(t_1, \ldots, t_r) \in K[t_1, \ldots, t_r]$ be the product of the leading coefficients of the $q_i(X)$. Then each $x_i$ is integral over the ring $K[t_1, \ldots, t_r]_f$.

Let $\overline{K}$ be an algebraic closure of $K$. Since $\overline{K}$ is infinite and $f \neq 0$, there exist $y_1, \ldots, y_r \in \overline{K}$ such that $f(y_1, \ldots, y_r) \neq 0$. Consider the unique homomorphism $\phi : K[t_1, \ldots, t_r] \to \overline{K}$ which is the identity on $K$, and for which $\phi(t_i) = y_i$. Since $\phi(f) \neq 0$, the universal property of localization shows that $\phi$ extends uniquely to a homomorphism $\phi : K[t_1, \ldots, t_r]_f \to \overline{K}$. By Theorem 2, we can extend $\phi$ to a homomorphism

$$\tilde{\phi} : K[t_1, \ldots, t_r]_f[x_1, \ldots, x_n] \to \overline{K}$$

which is the identity on $K$. But $K[t_1, \ldots, t_r]_f[x_1, \ldots, x_n] = L$. Since $L$ is a field, $\tilde{\phi}$ must be injective, so $L$ can be embedded as a subfield of $\overline{K}$ containing $K$. Thus $L/K$ is algebraic, contradiction. □

We now consider the following related question: given a subring $A$ of a field $K$, and a homomorphism $\phi$ from $A$ to an algebraically closed field $k$, when is it possible to extend $\phi$ to a larger subring of $K$? If $A$ is not integrally closed in $K$, then by Theorem 2 we can extend $\phi$ to the integral closure $\overline{A}$ of $A$ in $K$. Can we extend even further? The following result is the key to understanding the answer to this question:
**Theorem 5.** Let $A$ be a subring of a field $K$, and let $x \in K^\times$. If $\phi$ is a homomorphism from $A$ to an algebraically closed field $k$, then $\phi$ can be extended to a homomorphism from either $A[x]$ or $A[x^{-1}]$ to $k$.

**Proof.** By first extending $\phi$ to a homomorphism from $A_p$ to $k$, with $p = \ker(\phi)$, we may assume that $A$ is a local ring with maximal ideal $m = \ker(\phi)$.

If $x$ is integral over $A$, then $\phi$ has an extension to $A[x]$ by Theorem 2.

Now assume that $x$ is not integral over $A$, and let $B = A[x^{-1}]$. We claim that $mB \neq B$. Assuming this, it follows that $mB$ is contained in some maximal ideal $M$ of $B$. As $M \cap A$ contains $m$ and $m$ is maximal, it follows that $M \cap A = m$. Let $y$ be the image in $B/M$ of $x^{-1} \in B$. Since $B/M = (A/m)[y]$, we can extend the given map $\psi : A/m \to k$ to $\tilde{\psi} : B/M \to k$. Indeed, if $y$ is algebraic over $A/m$, then this follows from Theorem 1, while if $y$ is transcendental then we can define $\tilde{\psi}$ by sending $y$ to any element of $k$ that we like. (In fact, $y$ must be algebraic over $A/m$ by the Algebraic Nullstellensatz, but we don’t need to use this.) Composing with the natural projection $B \to B/M$ gives the desired extension.

To prove the claim, note that otherwise we can write

$$1 = a_0 + a_1 x^{-1} + \cdots + a_n x^{-n}$$

with $a_i \in m$. Multiplying by $x^n$, we get

$$(1 - a_0)x^n + b_{n-1}x^{n-1} + \cdots + b_0 = 0$$

with $b_i \in A$. As $a_0 \in m$, $1 - a_0 \in A^\times$, and dividing by $(1 - a_0)^{-1}$ we see that $x$ is integral over $A$, a contradiction. $\square$

**Corollary 1.** Let $A$ be a subring of a field $K$, and let $\phi$ be a homomorphism from $A$ to an algebraically closed field $k$. If $R$ is a maximal subring of $K$ for which $\phi$ extends to a homomorphism $\Phi : R \to k$ (such subrings exist by Zorn’s lemma), then $R$ is a valuation ring of $K$: for every $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$.

**Proof.** Let $\Sigma$ be the set of pairs $(C, \psi)$ where $C$ is a subring of $K$ and $\psi : C \to k$ is a homomorphism extending $\phi$. By Zorn’s lemma, there is a maximal element $(R, \Phi)$ of $\Sigma$, and $R$ is a valuation ring by Theorem 5. $\square$

Note that the ring $R$ given in Corollary 1 is a local ring with maximal ideal $m$ equal to $\ker(\Phi)$: otherwise we could extend $\Phi$ even further to $R_p$, where $p = \ker(\Phi)$. (Exercise: $R_p \supseteq R$ unless $R$ is already a local ring with maximal ideal $p$. Another exercise: Show that every valuation ring is a local ring.) Combining this observation with Corollary 1, we
obtain the following result (which is how the Extension Theorem is formulated in [Bump]):

**Theorem 6** (The Extension Theorem). *Let $A$ be a subring of a field $K$, and let $\phi : A \to k$ be a homomorphism from $A$ to an algebraically closed field $k$. Then there exists a valuation ring $R$ of $K$ containing $A$, and a homomorphism $\Phi : R \to k$ extending $\phi$, such that $\ker(\Phi)$ is the unique maximal ideal of $R$.*

One application of the Extension Theorem is the fact that if $A$ is a subring of a field $K$, then the integral closure of $A$ in $K$ is the intersection of all valuation rings of $K$ which contain $A$ (Problem 6 on Homework 1). The Extension Theorem can also be used to prove the extension theorem for integral extensions and the Going-Up Theorem, though in these notes we have gone the other way, and used those results to prove the Extension Theorem. See [Bump] for an alternative, more direct proof of the Extension Theorem. The main advantages to the proofs given in these notes are: (a) the statement of the Extension Theorem is more motivated, and (b) our approach to the extension theorem for integral extensions and the Going-Up Theorem does not require the assumption made in [Bump] that $A$ and $B$ are integral domains.