The game. Alice and Bob decide to play the following infinite game on the real number line. A subset $S$ of the unit interval $[0, 1]$ is fixed, and then Alice and Bob alternate playing real numbers. Alice moves first, choosing any real number $a_1$ strictly between 0 and 1. Bob then chooses any real number $b_1$ strictly between $a_1$ and 1. On each subsequent turn, the players must choose a point strictly between the previous two choices. Equivalently, if we let $a_0 = 0$ and $b_0 = 1$, then in round $n \geq 1$, Alice chooses a real number $a_n$ with $a_{n-1} < a_n < b_{n-1}$, and then Bob chooses a real number $b_n$ with $a_n < b_n < b_{n-1}$. Since a monotonically increasing sequence of real numbers which is bounded above has a limit (see [8, Theorem 3.14]), $\alpha = \lim_{n \to \infty} a_n$ is a well-defined real number between 0 and 1. Alice wins the game if $\alpha \in S$, and Bob wins if $\alpha \not\in S$.

Countable and uncountable sets. An set $X$ is called countable if it is possible to list the elements of $X$ in a (possibly repeating) infinite sequence $x_1, x_2, x_3, \ldots$. Equivalently, $X$ is countable if there is a function from the set $\{1, 2, 3, \ldots\}$ of natural numbers to $X$ which is onto. For example, every finite set is countable, and the set of natural numbers is countable. A set which is not countable is called uncountable. Cantor proved using his famous diagonalization argument that the real interval $[0, 1]$ is uncountable. We will give a different proof of this fact based on Alice and Bob’s game.

Proposition 1. If $S$ is countable, then Bob has a winning strategy.
Proof. Since $S$ is countable, one can enumerate the elements of $S$ as $s_1, s_2, s_3, \ldots$ Consider the following strategy for Bob. On move $n \geq 1$, he chooses $b_n = s_n$ if this is a legal move, and otherwise he randomly chooses any allowable number for $b_n$. Since $\alpha < b_n$ for all $n$, it follows that $\alpha \neq b_n$ for any $n \geq 1$, and thus $\alpha \not\in S$. This means that Bob always wins with this strategy!

If $S = [0,1]$, then clearly Alice wins no matter what either player does. Therefore we deduce:

**Corollary 1.** The interval $[0,1] \subset \mathbb{R}$ is uncountable.

This argument is in many ways much simpler than Cantor’s original proof!

**Perfect sets.** We now prove a generalization of the fact that $[0,1]$ is uncountable. This will also follow from an analysis of our game, but the analysis is somewhat more complicated. Given a subset $X$ of $[0,1]$, we make the following definitions:

- A *limit point* of $X$ is a point $x \in [0,1]$ such that for every $\epsilon > 0$, the open interval $(x - \epsilon, x + \epsilon)$ contains an element of $X$ other than $x$.

- $X$ is *closed* if every limit point of $X$ belongs to $X$.

- $X$ is *perfect* if it is non-empty, closed, and if every element of $X$ is a limit point of $X$.

For example, the famous middle-third Cantor set is perfect (see [8, §2.44]). If $L(X)$ denotes the set of limit points of $X$, then a nonempty set $X$ is closed $\iff L(X) \subseteq X$, and is perfect $\iff L(X) = X$. It is a well-known fact that every perfect set is uncountable (see [8, Theorem 2.43]). Using our infinite game, we will give a different proof of this fact. We recall the following basic property of the interval $[0,1]$:

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2Some authors consider the empty set to be perfect.
Every non-empty subset $X \subseteq [0, 1]$ has an infimum (or greatest lower bound), meaning that there exists a real number $\gamma \in [0, 1]$ such that $\gamma \leq x$ for every $x \in X$, and if $\gamma' \in [0, 1]$ is any real number with $\gamma' \leq x$ for every $x \in X$, then $\gamma' \leq \gamma$.

The infimum $\gamma$ of $X$ is denoted by $\inf(X)$.

Let’s say that a point $x \in [0, 1]$ is approachable from the right, denoted $x \in X^+$, if for every $\epsilon > 0$, the open interval $(x, x + \epsilon)$ contains an element of $X$. We can define approachable from the left (written $x \in X^-$) similarly using the interval $(x - \epsilon, x)$. It is easy to see that $L(X) = X^+ \cup X^-$, so that a non-empty set $X$ is perfect $\iff X = X^+ \cup X^-$.

The following two lemmas tell us about approachability in perfect sets.

**Lemma 1.** If $S$ is perfect, then $\inf(S) \in S^+$.

**Proof.** The definition of the infimum in (⋆) implies that $\inf(S)$ cannot be approachable from the left, so, being a limit point of $S$, it must be approachable from the right.

**Lemma 2.** If $S$ is perfect and $a \in S^+$, then for any $\epsilon > 0$, the open interval $(a, a + \epsilon)$ also contains an element of $S^+$.

**Proof.** Since $a \in S^+$, we can choose three points $x, y, z \in S$ with $a < x < y < z < a + \epsilon$. Since $(x, z) \cap S$ contains $y$, the real number $\gamma = \inf((x, z) \cap S)$ satisfies $x \leq \gamma \leq y$. If $\gamma = x$, then by definition (⋆) we have $\gamma \in S^+$. If $\gamma > x$, then definition (⋆) implies that $\gamma \in L(X)$ and $(x, \gamma) \cap S = \emptyset$, so that $\gamma \not\in S^-$ and therefore $\gamma \in S^+$.

From these lemmas, we deduce:

**Proposition 2.** If $S$ is perfect, then Alice has a winning strategy.

**Proof.** Alice’s only constraint on her $n$th move is that $a_{n-1} < a_n < b_{n-1}$. By induction, it follows from Lemmas 1 and 2 that Alice can always choose $a_n$ to be an element of $S^+ \subseteq S$. Since $S$ is closed, $\alpha = \lim a_n \in S$, so Alice wins!

From Propositions 1 and 2, we deduce:

**Corollary 2.** Every perfect set is uncountable.
Further analysis of the game. We know from Proposition 1 that Bob has a winning strategy if $S$ is countable, and it follows from Proposition 2 that Alice has a winning strategy if $S$ contains a perfect set. (Alice just chooses all of her numbers from the perfect subset.) What can one say in general? A well-known result from set theory [1, §6.2, Exercise 5] says that every uncountable Borel set\(^3\) contains a perfect subset. Thus we have completely analyzed the game when $S$ is a Borel set: Alice wins if $S$ is uncountable, and Bob wins if $S$ is countable. However, there do exist non-Borel uncountable subsets of $[0, 1]$ which do not contain a perfect subset [1, Theorem 6.3.7]. So we leave the reader with the following problem:

**Problem:** Do there exist uncountable subsets of $[0, 1]$ for which: (a) Bob has a winning strategy; (b) Alice does not have a winning strategy; or (c) neither Alice nor Bob has a winning strategy?

Related games. Our infinite game is a slight variant of the one proposed by Jerrold Grossman and Barry Turett in [2] (see also [6]). Propositions 1 and 2 above were motivated by parts (a) and (c), respectively, of their problem. The author originally posed Propositions 1 and 2 as challenge problems for the students in his Math 25 class at Harvard University in Fall 2000.

A related game (the “Choquet game”) can be used to prove the Baire category theorem (see §8.C of [5] and [3]). In Choquet’s game, played in a given metric space $X$, Pierre moves first by choosing a non-empty open set $U_1 \subseteq X$. Then Paul moves by choosing a non-empty open set $V_1 \subseteq U_1$. Pierre then chooses a non-empty open set $U_2 \subseteq V_1$, etc., yielding two decreasing sequences $U_n$ and $V_n$ of non-empty open sets with $U_n \supseteq V_n \supseteq U_{n+1}$ for all $n$, and $\cap U_n = \cap V_n$. Pierre wins if $\cap U_n = \emptyset$, and Paul wins if $\cap U_n \neq \emptyset$. One can show that if $X$ is complete, then Paul has a winning strategy, and if $X$ contains a non-empty open set $O$ which is a countable union of closed sets

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\(^3\)A Borel set is, roughly speaking, any subset of $[0, 1]$ that can be constructed by taking countably many unions, intersections, and complements of open intervals; see [8, §11.11] for a formal definition.
having empty interior, then Pierre has a winning strategy. As a consequence, one obtains the Baire category theorem: If $X$ is a complete metric space, then no open subset of $X$ can be a countable union of closed sets having empty interior.

Another related game is the Banach-Mazur game (see §6 of [7] and §8.H of [5]). A subset $S$ of the unit interval $[0,1]$ is fixed, and then Anna and Bartek alternate play. First Anna chooses a closed interval $I_1 \subseteq [0,1]$, and then Bartek chooses a closed interval $I_2 \subseteq I_1$. Next, Anna chooses a closed interval $I_3 \subseteq I_2$, and so on. Together the players’ moves determine a nested sequence $I_n$ of closed intervals. Anna wins if $\cap I_n$ has at least one point in common with $S$, otherwise Bartek wins. It can be shown that Bartek has a winning strategy if and only if $S$ is meagre (see Theorem 6.1 of [7]). (A subset of $X$ is called nowhere dense if the interior of its closure is empty, and is called meagre, or of the first category, if it is a countable union of nowhere dense sets.) It can also be shown, using the axiom of choice, that there exist sets $S$ for which the Banach-Mazur game is undetermined (neither player has a winning strategy).

For a more thorough discussion of these and many other topological games, we refer the reader to the survey article [9], which contains an extensive bibliography. Many of the games discussed in [9] are not yet completely understood.

Games like the ones we have been discussing play a prominent role in the modern field of descriptive set theory, most notably in connection with the axiom of determinacy (AD). (See Chapter 6 of [4] for a more detailed discussion.) Let $X$ be a given subset of the space $\omega^\omega$ of infinite sequences of natural numbers, and consider the following game between Alice and Bob. Alice begins by playing a natural number, then Bob plays another (possibly the same) natural number, then Alice again plays a natural number, and so on. The resulting sequence of moves determines an element $x \in \omega^\omega$. Alice wins if $x \in X$, and Bob wins otherwise. The axiom of determinacy states that this game is determined (i.e., one of the players has a winning strategy)
for every choice of $X$.

A simple construction shows that the axiom of determinacy is inconsistent with the axiom of choice. On the other hand, with Zermelo-Fraenkel set theory plus the axiom of determinacy (ZF+AD), one can prove many non-trivial theorems about the real numbers, including: (i) every subset of $\mathbb{R}$ is Lebesgue measurable; and (ii) every uncountable subset of $\mathbb{R}$ contains a perfect subset. Although ZF+AD is not considered a “realistic” alternative to ZFC (Zermelo-Fraenkel + axiom of choice), it has stimulated a lot of mathematical research, and certain variants of AD are taken rather seriously. For example, the axiom of projective determinacy is intimately connected with the continuum hypothesis and the existence of large cardinals (see [10] for details).

References


