The Prime Number Theorem

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1 The Riemann zeta function

The Prime Number Theorem asserts that the number of primes less than or equal to x is approximately equal to $\frac{x}{\log x}$ for large values of x (here and for the rest of these notes, log denotes the natural logarithm). This quantitative statement about the distribution of primes which was conjectured by several mathematicians (including Gauss) early in the nineteenth century, and was finally proved (independently) in 1896 by Hadamard and de la Vallée Poussin. Their proofs used fairly elaborate analytic methods. Many other proofs and generalizations of the prime number theorem have subsequently been found. In these lecture notes, we present a relatively simple proof of the Prime Number Theorem due to D. Newman (with further simplifications by D. Zagier). Our goal is to make the proof accessible for a reader who has taken a basic course in complex analysis but who does not necessarily have any background in number theory.

Throughout these notes, p will always represent a prime number. We begin with some definitions and notation.

Definition 1 *For $x \in \mathbb{R}$, $\pi(x)$ is the number of primes less than or equal to x.*

Definition 2 *Let $f, g : \mathbb{R} \to \mathbb{R}$. We say:*

- $f = O(g)$ *if there exists* $c \in \mathbb{R}$ such that $|f| \leq cg$
- $f \sim g$ *if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Using this notation, the Prime Number Theorem is the following statement:

Theorem 1 (Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\log x}.$$

We'll prove a large collection of auxiliary lemmas in order to establish this result, most of which will concern certain special meromorphic functions. The most important such function for our purposes is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is an exercise to show (using the Weierstrass M-test, for example) that for $\delta > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely and uniformly for $\mathrm{Re}(s) > 1 + \delta$, and therefore $\zeta(s)$ is analytic for $\mathrm{Re}(s) > 1$.

Why in the world should we be interested in the zeta function when we are trying to prove something about the distribution of prime numbers? The link is provided by Euler's product formula for the zeta function.

Lemma 1 *For* $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}. \tag{1}$$

PROOF. First note that the infinite product on the right-hand side of (1) converges absolutely for $\mathrm{Re}(s) > 1$, since

$$\sum_p |p^{-s}| = \sum_p p^{-\mathrm{Re}(s)} \le \sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)} < \infty.$$

(See Proposition 17.3 and Definition 17.4 in [Bak-Newman]).

Next, note that

$$\prod_p \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \frac{1}{1 - 5^{-s}} \cdots \tag{2}$$

By absolute convergence of the product, we can rewrite each term above using the relation $\frac{1}{1-x} = 1 + x + x^2 + \dots$, giving:

$$\prod_p \frac{1}{1-p^{-s}} = (1+2^{-s}+2^{-2s}+\dots)(1+3^{-s}+3^{-2s}+\dots)(1+5^{-s}+5^{-2s}+\dots)\dots$$

Now we expand the right-hand side of the above equation into a single infinite sum, each term a product of one summand from each sum. If infinitely many of these summands are not 1, their product will be zero. So all but finitely many nonzero summands contributing to each term will be 1, which means (by the unique factorization of natural numbers as products of primes) that

$$\prod_p^{\infty} \frac{1}{1-\frac{1}{p^{-s}}} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

■

Remark 1 *For an alternate proof, see pp. 238-239 in [Bak-Newman].*

Corollary 1 $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) > 1$.

Proof. This follows from the absolute convergence of the product $\prod_p \frac{1}{1-\frac{1}{p^s}}$ for $\mathrm{Re}(s) > 1$, which was established during the proof of the preceding lemma. ■

A crucial step for the proof of the Prime Number Theorem is to define a meromorphic continuation of the Riemann zeta function beyond its original domain of definition, and to study the zeros and poles of the resulting function.[1]

As a first step toward defining an extension of $\zeta(s)$, we have the following lemma:

Lemma 2

- *If* $\mathrm{Re}(s) > 1$, *then*

$$\int_1^{\infty} x^{-s} dx = \frac{1}{s-1}.$$

[1] In fact, $\zeta(s)$ extends to a meromorphic function on the entire complex plane, with a simple pole at $s = 1$ and no other poles, but we only need to extend $\zeta(s)$ to $\mathrm{Re}(s) > 0$ for the proof of the Prime Number Theorem.

- *If* $\mathrm{Re}(s) > 0$ *and* $n \in \mathbb{N}$, *then*

$$\int_n^x \frac{s}{u^{s+1}} du = \frac{1}{n^s} - \frac{1}{x^s}.$$

PROOF. Both parts of the lemma follow easily from the (complex version of the) fundamental theorem of calculus, and are left as exercises for the reader. [2] ■

We can now prove:

Lemma 3 *The function* $\zeta(s) - \frac{1}{s-1}$, *initially defined for* $\mathrm{Re}(s) > 1$, *extends to an analytic function on the half-plane* $\mathrm{Re}(s) > 0$.

PROOF. When $\mathrm{Re}(s) > 1$, we can use Lemma 2 to obtain

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\, dx. \tag{3}$$

We can rewrite the integral in (3) as a sum of integrals going from each integer to the next, and then combining the sums gives

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \int_n^{n+1} (\frac{1}{n^s} - \frac{1}{x^s})\, dx. \tag{4}$$

We claim that the series given by the right-hand side of (4) converges absolutely for $\mathrm{Re}(s) > 0$. To see this, we first use Lemma 2 to rewrite the integrand as an integral over a new variable u:

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| = \left| \int_n^{n+1} \int_n^x \frac{s\, du}{u^{s+1}}\, dx \right|. \tag{5}$$

We then bound the outer integral on the right-hand side of (5) using the M-L inequality. The length is 1, so we obtain

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leq \max_{n \leq x \leq n+1} \left| \int_n^x \frac{s\, du}{u^{s+1}}\, dx \right|. \tag{6}$$

[2]Note that if f is an analytic function on a neighborhood of the real interval $[a, b]$, then $\int_a^b f(x)dx = \int_C f(z)dz$, where C is the line segment from a to b.

We can bound the right-hand side of (6) by using M-L again, noting that the length is always at most one, so that

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leq \max_{n \leq u \leq n+1} \left| \frac{s}{u^{s+1}} \right| = \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

Our claim, and the lemma itself, now follows from the evident fact that the sum $\sum_{n=1}^{\infty} \frac{|s|}{n^{\mathrm{Re}(s)+1}}$ converges for $\mathrm{Re}(s) > 1$. ■

Corollary 2 *$\zeta(s)$ can be extended to a meromorphic function on $\mathrm{Re}(s) > 0$, with a simple pole of residue 1 at $s = 1$ and no other poles.*

■

Now that we have extended the domain of definition of $\zeta(s)$, we need some information on its zeros in the right half-plane $\mathrm{Re}(s) > 0$.[3] This is provided by the following two results. The following lemma is elementary but admittedly seems to come out of nowhere.

Lemma 4 *For all $x, y \in \mathbb{R}$ with $x > 1$, we have*

$$\left| \zeta^3(x)\zeta^4(x+iy)\zeta^2(x+2iy) \right| \geq 1. \tag{7}$$

Proof. Using Lemma 1, it suffices to prove that for each prime p, we have

$$\left| \left(1 - \frac{1}{p^x}\right)^3 \left(1 - \frac{1}{p^{x+iy}}\right)^4 \left(1 - \frac{1}{p^{x+2iy}}\right)^2 \right| \leq 1. \tag{8}$$

If we let $\frac{1}{p^x} = r$ and $\frac{1}{p^{iy}} = e^{i\theta}$, then $0 < r < 1$ and we can rewrite (8) as

$$\left| (1-r)^3(1-re^{i\theta})^4(1-re^{2i\theta})^2 \right| \leq 1. \tag{9}$$

It therefore suffices to prove the elementary fact that for all $0 \leq r < 1$ and all $\theta \in \mathbb{R}$, we have

$$\left| (1-re^{i\theta})^4(1-re^{2i\theta})^2 \right| \leq \frac{1}{(1-r)^3}. \tag{10}$$

[3] As we will see, $\zeta(s)$ has no zeros for $\mathrm{Re}(s) \geq 1$. The famous *Riemann hypothesis* is the assertion that all the zeros of $\zeta(s)$ in $\mathrm{Re}(s) > 0$ lie on the line $\mathrm{Re}(s) = 1/2$. This is considered by many people to be the most important unsolved problem in all of mathematics.

To establish (10), fix r with $0 < r < 1$, and let $f(\theta) = \left|(1 - re^{i\theta})^4(1 - re^{2i\theta})^2\right|$. A short calculation shows that $f(\theta) = (1 + r^2 - 2r\cos\theta)^2(1 + r^2 - 2r\cos 2\theta)$. Letting $u = cos\theta$ and using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we can rewrite this as

$$f(\theta) = g(u) := (1 + r^2 - 2ru)^2(1 + r^2 + 2r - 4ru^2).$$

We then can apply the arithmetic-geometric inequality

$$a^2 b \leq \left(\frac{2a + b}{3}\right)^3$$

with $a = 1 + r^2 - 2ru$ and $b = 1 + r^2 + 2r - 4ru^2$ to get

$$g(u) \leq h(u) := \frac{(3 + 3r^2 - 2r(2u^2 + 2u - 1))^3}{27}.$$

Basic calculus shows that $h(u)$ is maximized at $u = -\frac{1}{2}$ (where $a = b = 1 + r + r^2$), so

$$\max_\theta f(\theta) = g\left(-\frac{1}{2}\right) = (1 + r + r^2)^3 < (1 + r + r^2 + \dots)^3 = \frac{1}{(1 - r)^3},$$

which proves (10) as desired. ■

Lemma 5 $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) \geq 1$.

PROOF. It follows from Corollary 1 that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$, so it suffices to consider potential zeros with $\mathrm{Re}(s) = 1$.

Now, we consider the inequality given by equation (7). Suppose that $\zeta(s)$ has a zero at $s = 1 + iy_0$. We know that $\zeta(s)$ is analytic at $s = 1 + 2iy_0$ and has only a simple pole at $s = 1$, so that

$$\lim_{x \to 1^+} \zeta^3(x)\zeta^4(x + iy_0)\zeta^2(x + 2iy_0) = 0.$$

This contradicts inequality (7), proving the lemma. ■

2 The Prime Number Theorem

In addition to the function $\zeta(s)$, we're also going to be interested in a real function $\vartheta(x)$ defined by

$$\vartheta(x) = \sum_{p \le x} \log p.$$

The reason we care about ϑ is because of the following lemma:

Lemma 6 *The Prime Number Theorem holds if and only if $\vartheta(x) \sim x$.*

Proof. Since $\vartheta(x)$ has at most $\pi(x)$ summands, we have, for $x \ge 1$, the inequality

$$0 \le \vartheta(x) \le \pi(x) \log x.$$

Dividing by x, we obtain in particular that

$$\frac{\vartheta(x)}{x} \le \pi(x) \frac{\log x}{x}. \tag{11}$$

Also, for all $\epsilon > 0$, we have

$$\vartheta(x) \ge \sum_{x^{1-\epsilon} < p \le x} \log p, \tag{12}$$

since the primes between $x^{1-\epsilon}$ and x are a subset of all primes less than or equal to x. Bounding this sum from below, we get (by the definition of $\pi(x)$ and the monotonicity of the logarithm function) the inequality

$$\vartheta(x) \ge (1-\epsilon)(\log x)(\pi(x) - \pi(x^{1-\epsilon})).$$

Using the obvious fact that $\pi(x^{1-\epsilon}) \le x^{1-\epsilon}$, we obtain the inequality

$$\vartheta(x) \ge (1-\epsilon)(\log x)(\pi(x) - x^{1-\epsilon}),$$

which we can rewrite as

$$\pi(x) \le \frac{1}{1-\epsilon} \frac{\vartheta(x)}{\log x} + x^{1-\epsilon}. \tag{13}$$

Combining (11) and (13), we find that for every $\epsilon > 0$, we have

$$\frac{\vartheta(x)}{x} \le \pi(x) \frac{\log x}{x} \le \frac{1}{1-\epsilon} \frac{\vartheta(x)}{x} + \frac{\log x}{x^{\epsilon}}. \tag{14}$$

Since for each $\epsilon > 0$ we have $\frac{\log x}{x^\epsilon} \to 0$ as $x \to \infty$, it follows easily from (14) that $\frac{\vartheta(x)}{x} \to 1$ if and only if $\pi(x)\frac{\log x}{x} \to 1$. ■

We'll eventually prove the Prime Number Theorem by showing that $\vartheta(x) \sim x$, but for now we content ourselves with proving the weaker fact that $\vartheta(x)$ grows at most linearly with x. The idea for using the binomial coefficient $\binom{2n}{n}$ to prove this goes back to the Russian mathematician Chebyshev.

Lemma 7 *$\vartheta(x) = O(x)$.*

Proof. Let $n \in \mathbb{N}$. Then by the binomial theorem,

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots + + \binom{2n}{2n} \geq \binom{2n}{n}.$$

But

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \geq \prod_{n<p\leq 2n} p,$$

since $\binom{2n}{n}$ is an integer and no prime greater than n can divide $n!$. Therefore we have

$$2^{2n} \geq \prod_{n<p\leq 2n} p = e^{\vartheta(2n)-\vartheta(n)}, \tag{15}$$

where the last equality follows from the definition of ϑ. Taking logarithms in (15) gives

$$\vartheta(2n) - \vartheta(n) \leq 2n \log 2.$$

It follows that for $m \in \mathbb{N}$, we have

$$\begin{aligned}\vartheta(2^m) = \textstyle\sum_{n=1}^m \left(\vartheta(2^n) - \vartheta(2^{n-1})\right) \leq \sum_{n=1}^m 2^n \log 2 &= (2^{m+1} - 2)\log 2 \\ &< 2^{m+1} \log 2.\end{aligned}$$

Now for any $x \geq 1$, we can choose m with $2^{m-1} \leq x < 2^m$, and then

$$\vartheta(x) \leq \vartheta(2^m) \leq 2^{m+1} \log 2 = (4 \log 2) 2^{m-1} \leq (4 \log 2) x,$$

so that $\vartheta(x) = O(x)$, as needed. ■

In order to show that $\vartheta(x) \sim x$, we will also need the following two lemmas. (Note that the function $\vartheta(x)$ is obviously non-decreasing).

Lemma 8 *Let* $\psi(x) : \mathbb{R} \to \mathbb{R}$. *Then* $\psi(x) \sim x$ *if and only if for every* $\epsilon > 0$ *we have*

$$(1-\epsilon)x < \psi(x) < (1+\epsilon)x$$

for all sufficiently large x.

PROOF. Exercise. ■

Lemma 9 *If* $\psi(x) : [1,\infty) \to \mathbb{R}$ *is any non-decreasing function of* x *such that the improper integral* $\int_1^\infty \frac{\psi(u)-u}{u^2} du$ *converges, then* $\psi(x) \sim x$.

PROOF. First, assume there is some $\epsilon > 0$ such that $\psi(x) \geq (1+\epsilon)x$ for arbitrarily large x. Since ψ is non-decreasing, we have for such x the chain of inequalities

$$\begin{aligned} \int_x^{(1+\epsilon)x} \frac{\psi(u)-u}{u^2} du = \int_x^{(1+\epsilon)x} \left(\frac{\psi(u)}{u} - 1 \right) \frac{du}{u} &\geq \int_x^{(1+\epsilon)x} \left(\frac{(1+\epsilon)x}{u} - 1 \right) \frac{du}{u} \\ &= \int_1^{1+\epsilon} \left(\frac{1+\epsilon}{w} - 1 \right) \frac{dw}{w} = c \end{aligned}$$

for some constant $c > 0$ depending only on ϵ (and not on x). (Note that we have made the change of variables $w = u/x$).

Since the inequality

$$\int_x^{(1+\epsilon)x} \frac{\psi(u)-u}{u^2} du \geq c$$

holds for arbitrarily large values of x, it follows that the improper integral $\int_1^\infty \frac{\psi(u)-u}{u^2}$ cannot converge, a contradiction. By an analogous argument, we also obtain a contradiction if there is some $\epsilon > 0$ such that $\psi(x) \leq (1-\epsilon)x$ for arbitrarily large x. It therefore follows by Lemma 8 that $\psi(x) \sim x$ as desired. ■

Making a simple change of variables, we obtain:

Corollary 3 *If the improper integral*

$$\int_0^\infty \left(\vartheta(e^t)e^{-t} - 1 \right) dt \tag{16}$$

converges, then the Prime Number Theorem is true.

PROOF. Making the change of variables $u = e^t$, we have

$$\int_1^\infty \frac{\vartheta(u) - u}{u^2}\, du = \int_0^\infty \frac{\vartheta(e^t) - e^t}{e^t}\, dt = \int_0^\infty \left(\vartheta(e^t)e^{-t} - 1\right)\, dt.$$

Now apply Lemmas 6 and 9. ■

For notational convenience, we define the function $H(t) : [0, \infty) \to \mathbb{R}$ to be the integrand in (16), i.e.,

$$H(t) := \vartheta(e^t)e^{-t} - 1.$$

For future use, we note the following about the function H(t):

Lemma 10 *The function $H(t)$ is bounded and piecewise continuous.*

PROOF. The piecewise continuity of H follows from that of ϑ, and the boundedness of H is equivalent to Lemma 7. ■

3 The Laplace transform

In the previous section, we showed that in order to prove the Prime Number Theorem, it is enough to establish the convergence of the improper integral $\int_0^\infty H(t)dt$. However, it is difficult to establish the convergence of this integral directly. We therefore proceed by a rather indirect method involving the *Laplace transform* of $H(t)$.

Definition 3 *Let $h(t) : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function. The* Laplace transform *of $h(t)$ is the complex function $(\mathcal{L}h)(s)$ given by*

$$(\mathcal{L}h)(s) = \int_0^\infty e^{-st}h(t)dt$$

wherever the improper integral converges.

Lemma 11 *If there exist real constants B, C such that $|h(t)| \leq Ce^{Bt}$ for $0 \leq t < \infty$, then the improper integral defining the Laplace transform of $h(t)$ converges absolutely and defines an analytic function for $\mathrm{Re}(s) > B$.*

PROOF. We have (setting $\sigma = \mathrm{Re}(s)$):

$$\int_0^T |e^{-st}h(t)|dt \leq C \int_0^T e^{-(\sigma-B)t}dt = \frac{C}{\sigma - B}(1 - e^{-(\sigma-B)T}).$$

Letting $T \to \infty$ and using the fact that $\sigma - B > 0$, we obtain the estimate

$$\int_0^T |e^{-st}h(t)|dt \leq \frac{C}{\sigma - B}. \tag{17}$$

The result follows easily from (17). ■

There are many useful (and easy to prove) formulas connected to the Laplace transform. For example:

Lemma 12 *Let $g(t), h(t) : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions with $|h(t)| \leq Ce^{Bt}$ for $0 \leq t < \infty$. Then:*

- $\mathcal{L}(g + h) = \mathcal{L}g + \mathcal{L}h$.
- *If $g(t) = e^{at}h(t)$, then $(\mathcal{L}g)(s) = (\mathcal{L}h)(s - a)$ for $\mathrm{Re}(s) > B + a$.*
- $\mathcal{L}(1)(s) = \frac{1}{s}$.

PROOF. Exercise for the reader. ■

In order to compute the Laplace transform of $H(t)$, we first define the function $\Phi(s)$ on the half-plane $\mathrm{Re}(s) > 1$ by the formula

$$\Phi(s) = \sum_p \frac{\log p}{p^s}.$$

For every $\delta > 0$, it is easy to show that $\sum_{n=1}^{\infty} \frac{\log n}{n^s}$ converges absolutely and uniformly for $\mathrm{Re}(s) > 1 + \delta$, and therefore $\Phi(s)$ is analytic for $\mathrm{Re}(s) > 1$ by the usual argument.

Lemma 13 $(\mathcal{L}\vartheta(e^t))(s) = \frac{\Phi(s)}{s}$ *for* $\mathrm{Re}(s) > 1$.

PROOF. By Lemma 7, there exists a constant C such that $\vartheta(e^t) \leq Ce^t$ for all $t \geq 0$. By Lemma 11, the integral defining $(\mathcal{L}\vartheta(e^t))$ converges absolutely for $\mathrm{Re}(s) > 1$.

Let p_n denote the nth prime number, so that $p_1 = 2, p_2 = 3, \ldots$, and for convenience define $p_0 = 0$. Then $\vartheta(e^t)$ is constant for $\log p_n < t < \log p_{n+1}$, so

$$\int_{\log p_n}^{\log p_{n+1}} e^{-st}\vartheta(e^t)dt = \vartheta(p_n)\int_{\log p_n}^{\log p_{n+1}} e^{-st}dt = \frac{1}{s}\vartheta(p_n)(p_n^{-s} - p_{n+1}^{-s}).$$

Summing over all $n \geq 1$ and using the obvious formula $\vartheta(p_n) - \vartheta(p_{n-1}) = \log p_n$, we obtain

$$\begin{aligned}\int_0^\infty e^{-st}\vartheta(e^t)dt = \tfrac{1}{s}\sum_{n=1}^\infty \vartheta(p_n)(p_n^{-s} - p_{n+1}^{-s}) &= \tfrac{1}{s}\sum_{n=1}^\infty \left(\vartheta(p_n) - \vartheta(p_{n-1})\right) p_n^{-s} \\ &= \tfrac{1}{s}\sum_{n=1}^\infty \tfrac{\log p_n}{p_n^s} = \tfrac{1}{s}\Phi(s)\end{aligned}$$

for $\mathrm{Re}(s) > 1$. ■

Corollary 4

$$(\mathcal{L}H)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$$

for $\mathrm{Re}(s) > 0$.

Proof. This follows immediately from Lemmas 12 and 13. ■

The following two lemmas provide crucial analytic information about the function $\Phi(s)$. The key point is that the sum of $\Phi(s)$ and the logarithmic derivative of $\zeta(s)$ extends to an analytic function for $\mathrm{Re}(s) > 1/2$.

Lemma 14 *The function* $\Phi(s) - \frac{1}{(s-1)}$, *initially defined for* $\mathrm{Re}(s) > 1$, *extends to a meromorphic function on the half-plane* $\mathrm{Re}(s) > 1/2$, *and is analytic for* $\mathrm{Re}(s) \geq 1$.

Proof. We logarithmically differentiate both sides of Euler's product formula for the zeta function and multiply by -1, giving (for $\mathrm{Re}(s) > 1$)

$$\begin{aligned}-\tfrac{\zeta'(s)}{\zeta(s)} &= -(\log(\textstyle\prod_p \tfrac{1}{1-p^{-s}}))' \\ &= \textstyle\sum_p (\log(1 - p^{-s}))' \\ &= \textstyle\sum_p \tfrac{p^{-s}\log p}{1-p^{-s}} = \sum_p \tfrac{\log p}{p^s - 1},\end{aligned}$$

where the term-by-term differentiation is justified by the absolute convergence of the sum.

Using the identity

$$\frac{1}{p^s-1} = \frac{1}{p^s} - \frac{1}{p^s(p^s-1)},$$

we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}. \tag{18}$$

The infinite sum appearing in (18) converges to an analytic function for $\mathrm{Re}(s) > \frac{1}{2}$ by comparison with $\sum_{n=1}^{\infty} \frac{\log n}{n^{2s}}$. The lemma now follows from the fact that $\frac{\zeta'(s)}{\zeta(s)}$ is meromorphic for $\mathrm{Re}(s) > 0$ by Lemma 3, and $\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic for $\mathrm{Re}(s) \geq 1$ by Lemmas 3 and 5.[4] ■

Making a change of variables, we find:

Corollary 5 $(\mathcal{L}H)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ *extends to a meromorphic function on the half-plane* $\mathrm{Re}(s) > -1/2$*, and is analytic for* $\mathrm{Re}(s) \geq 0$.

PROOF. This follows easily from the fact that the residue of $\frac{\Phi(s+1)}{s+1}$ at $s = 0$ is 1. ■

Remark 2 *We emphasize that the analyticity of* $\frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ *on the line* $\mathrm{Re}(s) = 0$*, which is one of the key ingredients in the proof of Lemma 15 below, comes from the subtle non-vanishing result established in Lemma 5.*

We now state the key analytic result necessary for our proof of the Prime Number Theorem, which deals with analytic continuations of Laplace transforms. The proof of this result, which uses the Cauchy integral formula, will be given in the next section.

Theorem 2 (Analytic theorem) *Let* $f(t)$ *for* $t \geq 0$ *be a bounded and piecewise continuous function such whose Laplace transform* $g(s) = \int_0^\infty f(t)e^{-st}\,dt$*, initially defined for* $\mathrm{Re}(s) > 0$*, extends to an analytic function for* $\mathrm{Re}(s) \geq 0$*. Then the improper integral* $\int_0^\infty f(t)\,dt$ *converges, and its value is* $g(0)$*.*

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[4]We are using here the basic fact that if f is meromorphic, then the poles of f'/f occur precisely at the zeros and poles of f. Note that the Riemann hypothesis implies that (except for the simple pole at $s = 1$), $\Phi(s)$ is in fact analytic for $\mathrm{Re}(s) > 1/2$.

Remark 3 *The subtlety here is that a priori, the fact that $(\mathcal{L}f)(s)$ can be extended to an analytic function in a neighborhood of $s = 0$ does not obviously imply that the integral used to define $(\mathcal{L}f)(s)$ for $\mathrm{Re}(s) > 0$ converges at $s = 0$. In fact, one needs the stronger hypothesis of the theorem, namely that $(\mathcal{L}f)(s)$ extends across the entire imaginary axis, to reach this non-obvious conclusion.*

Assuming the analytic theorem, let's see how to use it to prove the Prime Number Theorem.

Lemma 15 *The improper integral $\int_0^\infty H(t)\,dt$ converges.*

Proof. By Lemma 10, $H(t)$ is piecewise continuous and bounded. Applying Theorem 2 to $H(t)$, whose Laplace transform $(\mathcal{L}H)(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ extends to an analytic function for $\mathrm{Re}(s) \geq 0$ by Corollary 5, we conclude that $\int_0^\infty H(t)\,dt$ converges, as desired. ■

Putting together Corollary 3 and Lemma 15 completes the proof the Prime Number Theorem (modulo Theorem 2). We give the proof of Theorem 2, which is all we have left to establish, in the next section.

4 Proof of the Analytic Theorem

In this section, we complete the proof of the Prime Number Theorem by using complex analytic techniques to prove the analytic theorem about Laplace transforms from the previous section.

Here is the proof of Theorem 2, following Newman and Zagier. We now use the variable z instead of s in order to keep the notation more familiar.

Consider the sequence of functions $g_T(z) = \int_0^T f(t)e^{-zt}\,dt$. These functions are all entire, as follows from Morera's theorem (see Theorem 17.9 in [Bak-Newman]). We are trying to show that $\lim_{T\to\infty} g_T(0)$ exists and equals $g(0)$.

Choose a large real number R, and consider the curve γ (oriented counterclockwise) which is the boundary of the region $\{z \in \mathbb{C} \mid |z| \leq R, \mathrm{Re}(z) > -\delta\}$, where we choose $\delta > 0$ small enough (depending on R) so that $g(z)$ is analytic on and inside γ. (Such a δ exists by compactness and the fact that $g(z)$ is analytic for $\mathrm{Re}(z) \geq 0$.)

Since the function $G_T(z) := (g(z) - g_T(z))e^{zT}\left(1 + \frac{z^2}{R^2}\right)$ is analytic on and inside γ, and $G_T(0) = g(0) - g_T(0)$, we can apply the Cauchy integral formula to obtain

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_\gamma (g(z) - g_T(z))e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}. \tag{19}$$

We now analyze the contributions to the integral in (19) from the different pieces of the curve γ.

Let γ_+ be the semicircle $\{|z| = R, \mathrm{Re}(z) \geq 0\}$. We wish to bound the absolute value of the integrand $G_T(z)/z$ on γ_+. We claim that we can bound it by $\frac{2B}{R^2}$, where we let $B = \max_{t\geq 0}|f(t)|$. To see this, first note that

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right| \leq B\int_T^\infty \left|e^{-zt}\right|\,dt = \frac{Be^{-\mathrm{Re}(z)T}}{\mathrm{Re}(z)}$$

for $\mathrm{Re}(z) > 0$. Also, we have

$$\begin{aligned}\left|e^{zT}(1 + \tfrac{z^2}{R^2})\tfrac{1}{z}\right| &= e^{\mathrm{Re}(z)T}\tfrac{R^2+z^2}{R^2z}\\ &= e^{\mathrm{Re}(z)T}\tfrac{z(\overline{z}+z)}{R^2z}\\ &= e^{\mathrm{Re}(z)T}\tfrac{2\mathrm{Re}(z)}{R^2}.\end{aligned}$$

Together, these estimates establish our claim that $|G_T(z)| \leq \frac{2B}{R^2}$ for $z \in \gamma_+$. The ML-inequality now gives the estimate

$$\left|\frac{1}{2\pi i}\int_{\gamma_+}(g(z) - g_T(z))e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{B}{R}. \tag{20}$$

For the remainder $\gamma_- = \gamma \cap \{\mathrm{Re}(z) < 0\}$ of the curve γ, we'll consider the functions $g(z)$ and $g_T(z)$ separately.

Since $g_T(z)$ is entire, we can replace the contour γ_- of integration in the integral involving $g_T(z)$ by the semicircle $\gamma'_- = \{\mathrm{Re}(z) < 0, |z| = R\}$. The same estimate as before gives

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}\,dt\right| \leq B\int_0^T e^{-\mathrm{Re}(z)t}\,dt < \frac{Be^{-\mathrm{Re}(z)T}}{|\mathrm{Re}(z)|}$$

for $\mathrm{Re}(z) < 0$. The other estimates go through as well, so that

$$\left|\frac{1}{2\pi i}\int_{\gamma_-} g_T(z)e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{B}{R}. \tag{21}$$

Finally, we observe that the integrand $g(z)e^{zT}\frac{\left(1+\frac{z^2}{R^2}\right)}{z}$ converges to zero uniformly on compact sets as $T \to \infty$. Indeed, the integrand is the product of $g(z)\frac{\left(1+\frac{z^2}{R^2}\right)}{z}$, which is independent of T, and e^{zT}, which goes to zero uniformly on compact subsets of γ_-. Therefore

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma_-} g(z)e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = 0. \tag{22}$$

It now follows from (19)–(22) that

$$\limsup_{T\to\infty} |g(0) - g_T(0)| \leq \frac{2B}{R}. \tag{23}$$

If we let $R \to \infty$[5], we conclude from (23) that

$$\lim_{T\to\infty} g_T(0) = g(0)$$

as needed. ■

Exercise 1 *Go through the proof of Theorem 2 and figure out what motivated the choice of the complicated-looking function*

$$G_T(z) := (g(z) - g_T(z))e^{zT} \left(1 + \frac{z^2}{R^2}\right).$$

Exercise 2 *Fill in the details in the following sketch of an alternative proof of Lemma 5: If $\zeta(s)$ has a zero of order μ at $s = 1 + iy$ and a zero of order ν at $s = 1 + 2iy$, then*

$$\lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon) = 1, \lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon \pm iy) = -\mu, \lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon \pm 2iy) = -\nu.$$

The inequality

$$\sum_{r=-2}^{2} \binom{4}{2+r} \Phi(1+\epsilon+iry) = \sum_p \frac{\log p}{p^{1+\epsilon}} (p^{iy/2} + p^{-iy/2})^4 \geq 0$$

then implies that $6 - 8\mu - 2\nu \geq 0$, so that $\mu = 0$.

[5]We are using the fact that B does not depend on R, which is why we need the assumption that f is bounded.

References

[1] J. Bak and D. Newman, *Complex Analysis* (2nd ed.), Springer UTM, Springer–Verlag, New York and Berlin, 1999.

[2] T. Gamelin, *Complex Analysis*, Springer–Verlag, New York and Berlin, 2001.

[3] D. Zagier, *Newman's short proof of the prime number theorem*, American Math. Monthly, October 1997, pp. 705-708.