REDUCED DIVISORS ON GRAPHS

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ABSTRACT. We give a new proof of the fact that given a reference vertex q and a divisor D on a graph G, there exists a unique q-reduced divisor equivalent to D.

1. Graphs

Let G be a graph[1], let $V(G)$ be the set of vertices of G, and fix a vertex $q \in V(G)$. For a non-empty subset S of $V(G)$ and $v \in S$, we denote by $\text{outdeg}_S(v)$ the number of edges connecting v to vertices not in S.

Let $\text{Div}(G)$ be the free abelian group on $V(G)$. Elements of $\text{Div}(G)$ are called *divisors* on G, and can be written as $D = \sum_{v \in V(G)} D(v)(v)$, where each $D(v) \in \mathbb{Z}$. Following [1], a divisor D on G is called q-*reduced*[2] if $D(v) \geq 0$ for all $v \in V(G) \backslash \{q\}$, and for every non-empty subset S of $V(G) \backslash \{q\}$, there exists a vertex $v \in S$ such that $D(v) < \text{outdeg}_S(v)$.

A *rational function* on G is a function $f : V(G) \to \mathbb{Z}$. We denote by $\mathcal{M}(G)$ the group of all rational functions on G. To divisors D and D' are called *equivalent* if $D - D' = \Delta(f)$ for some $f \in \mathcal{M}(G)$, where Δ denotes the Laplacian operator $\Delta : \mathcal{M}(G) \to \text{Div}(G)$ defined by

$$\Delta(f)(v) = \sum_{e=vw \text{ in } E(G)} (f(v) - f(w)) .$$

A divisor D is *effective* if $D(v) \geq 0$ for all $v \in V(G)$, and *effective outside* q if $D(v) \geq 0$ for all $v \in V(G) \backslash \{q\}$. We denote by $|D|$ the set of all effective divisors equivalent to D. Since equivalent divisors have the same degree, $|D|$ is always a finite set.

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[1] We assume our graphs to be connected and to have no loop edges, though multiple edges between vertices are allowed.

[2] This is equivalent to saying that the function $V(G) \backslash \{q\} \to \mathbb{Z}$ given by $v \mapsto D(v)$ is a *G-parking function relative to q*.

The following lemma gives a useful alternate characterization of q-reduced divisors.

Lemma 1.1. *A divisor D is q-reduced if and only if D is effective outside q, but for every non-constant function $f \in \mathcal{M}(G)$ having a global maximum at q, the divisor $D + \Delta(f)$ is not effective outside q.*

Proof. If D is q-reduced and $f \in \mathcal{M}(G)$ is a non-constant function with a global maximum at q, let S be the set of vertices where f attains its minimal value. By hypothesis, S is a non-empty subset of $V(G)\backslash\{q\}$, so there exists $v \in S$ such that $D(v) < \text{outdeg}_S(v)$. Since $\Delta(f)(v) <= -\text{outdeg}_S(v)$, it follows that $(D + \Delta(f))(v) < 0$, and thus $D + \Delta(f)$ is not effective outside q.

Conversely, suppose that D is effective outside q, but for every non-constant function $f \in \mathcal{M}(G)$ having a global maximum at q, the divisor $D + \Delta(f)$ is not effective outside q. Let S be any non-empty subset of $V(G)\backslash\{q\}$, let T be the complement of S, and let $f = \chi_T$ be the characteristic function of T, so that $f(v) = 1$ for $v \notin S$ and $f(v) = 0$ for $v \in S$. Then $\Delta(f)(v) = -\text{outdeg}_S(v)$ for $v \in S$, and $\Delta(f)(v) = \text{outdeg}_T(v)$ for $v \notin S$. Since f is a non-constant rational function having a global maximum at q, there exists a vertex $v \in V(G)$ such that $(D + \Delta(f))(v) < 0$. It follows from the formula for $\Delta(f)$ that $v \in S$ and $D(v) < \text{outdeg}_S(v)$. Thus D is q-reduced. □

List the vertices of G as $v_1, \ldots, v_n$, with q corresponding to v_j. Let $\delta_q \in \mathbb{Z}^n$ denote the vector with a 1 in the j^{th} coordinate and 0's elsewhere, and choose a strict probability vector $\mu \in \mathbb{Q}^n$, i.e., a vector with positive rational coordinates $\mu_i > 0$ such that $\sum \mu_i = 1$. (For example, μ could be the vector with a $1/n$ in each coordinate.) Let Q be the Laplacian matrix of G. Because G is connected, it is well-known that the kernel of Q, thought of as a linear transformation from $\mathbb{Q}^n$ to $\mathbb{Q}^n$, is spanned by μ, and the image of Q is the orthogonal complement of μ (with respect to the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{Q}^n$). Thus there is a unique vector $h \in \mathbb{Q}^n$ such that

$$Qh = \mu - \delta_q \text{ and } h_j = 0,$$

where h_j denotes the j^{th} coordinate of the vector h.

Identifying divisors on G with vectors in $\mathbb{Z}^n$ in the obvious way, we can define a functional $\hat{h} : \text{Div}(G) \to \mathbb{Q}$ by

$$\hat{h}(D) = \langle D, h \rangle.$$

Remark 1.2. Since $h_j = 0$, it follows that $h_i \geq 0$ for all i, and thus that $\hat{h}(D) \geq 0$ for all effective divisors D. Indeed, identifying h with a function from $V(G)$ to $\mathbb{Q}$, it follows from the maximum principle for graphs that the non-constant function h attains its minimum value at a vertex v_i where $\Delta(h)(v_i) < 0$, i.e., where $(Qh)_i < 0$. The only such vertex is $q = v_j$.

Lemma 1.3. *If D is an effective divisor on G and $\hat{h}(D) \leq \hat{h}(D')$ for all effective divisors D' equivalent to D, then D is q-reduced.*

Proof. Let f be a non-constant rational function on G having a global maximum at q, and let $D' = D + \Delta(f)$. Identifying f with a vector in $\mathbb{Z}^n$ in the obvious way, we have:

$$\begin{aligned}\hat{h}(D') &= \langle D + \Delta(f), h\rangle \\ &= \langle D, h\rangle + \langle Qf, h\rangle \\ &= \hat{h}(D) + \langle f, Qh\rangle \\ &= \hat{h}(D) + \left(\sum_{i=1}^{n} \mu_i f(v_i)\right) - f(q) \\ &< \hat{h}(D).\end{aligned}$$

By hypothesis, D' cannot be effective. It follows from Lemma 1.1 that D is q-reduced. ☐

We also have the following result from [1]:

Lemma 1.4. *Every divisor D on G is equivalent to a divisor which is effective outside q.*

Proof. Since the group $\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is finite, for each $v \in V(G)$ there is an integer $m_v \geq 1$ such that the divisor $P_v = m_v(v) - m_v(q)$ is principal. The result follows easily by adding appropriate positive integer multiples of each P_v to D.

Here is an alternate proof, which generalizes better to the metric graph situation. It suffices to construct a principal divisor P which is *strictly effective outside* q, meaning that $P(v) \geq 1$ for all $v \neq q$. Let k be the maximal distance between q and any vertex of G, and for $i = 0, \ldots, k$, let S_i be the set of vertices of G of distance i from q. Thus $S_0 = \{q\}$, S_1 is the set of neighbors of q, and $V(G)$ is the disjoint union of $S_0, \ldots, S_k$. For $i = 1, \ldots, k$, let f_i be the rational function which is 0 on $S_0 \cup \cdots \cup S_{i-1}$ and 1 on $T_i := S_i \cup \cdots \cup S_k$, and set

$$P_i = \Delta(f_i) = \sum_{v \in S_i} a_v(v) - \sum_{v' \in S_{i-1}} b_{v'}(v'),$$

where for $v \in S_i$, $a_v = \mathrm{outdeg}_{T_i}(v) \geq 1$ and $b_v = \mathrm{outdeg}_{S_i}(v) \geq 1$. Choose positive integers $c_k, c_{k-1}, \ldots, c_1$ inductively so that $c_k = 1$ and

$c_i a_v > c_{i+1} b_v$ for all $v \in S_i$ $(i = k-1, \ldots, 1)$. Then $P = \sum_{i=1}^{k} c_i P_i$ is strictly effective outside q. $\square$

As a consequence of Lemmas 1.3 and 1.4, we obtain the following result, which was originally proved in [1] by a different method:

Theorem 1.5. *If $D \in \mathrm{Div}(G)$, then there is a unique q-reduced divisor equivalent to G.*

Proof. We first prove the existence of a q-reduced divisor equivalent to D. By Lemma 1.4, we may assume that D is effective outside q. Then, replacing D by $D - k(q)$ if $k = D(q) < 0$, we may assume without loss of generality that D is effective. Since $|D|$ is a finite set, the functional $\hat{h} : |D| \to \mathbb{Q}$ achieves its minimum value at some effective divisor $D_q \in |D|$. By Lemma 1.3, D_q is q-reduced. This proves the existence part of the theorem.

For the uniqueness assertion, suppose D and D' are distinct q-reduced divisors with D equivalent to D'. Write $D' = D + \Delta(f)$ with $f \in \mathcal{M}(G)$ non-constant. Since D' is effective outside q and D is q-reduced, it follows from Lemma 1.1 that f does not have a global maximum at q. Thus the set S of vertices at which f attains its maximum value is a non-empty subset of $V(G)\backslash\{q\}$. Note that $\Delta(f)(v) \geq \mathrm{outdeg}_S(v)$ for all $v \in S$. Since D' is q-reduced, there is a vertex $v \in S$ for which $D'(v) < \mathrm{outdeg}_S(v)$. Since $D(v) \geq 0$, we have

$$D'(v) < \mathrm{outdeg}_S(v) \leq \Delta(f)(v) \leq (D + \Delta(f))(v) = D'(v),$$

a contradiction. $\square$

References

[1] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

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