

1. (a) (10 points) Compute the definite integral

$$\int_e^{e^2} \frac{1}{x\sqrt{\ln x}} dx.$$

- (b) (15 points) Find the indefinite integral

$$\int x^2 \sin(1-x) dx.$$

Solution:

- (a) Make a u -substitution with $u = \ln x$ and $du = (1/x) dx$ so that

$$\int_e^{e^2} \frac{1}{x\sqrt{\ln x}} dx = \int_1^2 \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_1^2 = 2(\sqrt{2} - 1).$$

- (b) Integrate by parts with $u = x^2$ and $dv = \sin(1-x) dx$ so that $du = 2x dx$ and $v = \cos(1-x)$, and

$$\int x^2 \sin(1-x) dx = x^2 \cos(1-x) - 2 \int x \cos(1-x) dx.$$

Integrate by parts with $u = x$ and $dv = \cos(1-x) dx$ so that $du = dx$ and $v = -\sin(1-x)$, and

$$\int x \cos(1-x) dx = -x \sin(1-x) + \int \sin(1-x) dx = -x \sin(1-x) + \cos(1-x) + C.$$

Thus,

$$\begin{aligned} \int x^2 \sin(1-x) dx &= x^2 \cos(1-x) - 2(-x \sin(1-x) + \cos(1-x) + C) \\ &= (x^2 - 2) \cos(1-x) + 2x \sin(1-x) + C. \end{aligned}$$

2. (a) (10 points) Find the indefinite integral

$$\int \sec^2 \theta \sin^3 \theta \, d\theta.$$

- (b) (15 points) Find the definite integral

$$\int_{1/e}^1 \ln x \, dx.$$

Solution:

- (a) Replace $\sin^2 \theta = 1 - \cos^2 \theta$:

$$\begin{aligned} \int \sec^2 \theta \sin^3 \theta \, d\theta &= \int \frac{1}{\cos^2 \theta} (1 - \cos^2 \theta) \sin \theta \, d\theta \\ &= \int \left(\frac{1}{\cos^2 \theta} - 1 \right) \sin \theta \, d\theta \\ &= \frac{1}{\cos \theta} + \cos \theta + C. \end{aligned}$$

- (b) Integrate by parts with $u = \ln x$ and $dv = dx$. In this case, $du = (1/x) dx$ and $v = x$.

$$\begin{aligned} \int_{1/e}^1 \ln x \, dx &= x \ln x \Big|_{1/e}^1 - \int_{1/e}^1 1 \, dx \\ &= -(1/e) \ln(1/e) - [1 - (1/e)] \\ &= 1/e - 1 + 1/e \\ &= 2/e - 1. \end{aligned}$$

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3. (25 points) Compute the indefinite integral

$$\int \frac{x^3}{x^2 - 2x + 1} dx.$$

Solution: First divide $x^3 \div (x^2 - 2x + 1) = x + 2 + (3x - 2)/(x - 1)^2$. Use partial fractions:

$$\frac{a}{x - 1} + \frac{b}{(x - 1)^2} = \frac{ax - a + b}{(x - 1)^2}.$$

$a = 3$ and $b = 1$. Therefore,

$$\begin{aligned}\int \frac{x^3}{x^2 - 2x + 1} dx. &= \frac{x^2}{2} + 2x + \int \left(\frac{3}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \frac{x^2}{2} + 2x + 3 \ln|x - 1| - \frac{1}{x - 1} + C.\end{aligned}$$

4. (25 points) Compute the improper integral

$$\int_{-\infty}^{\infty} \frac{4}{x^2 + 16} dx.$$

Solution: Dividing the numerator and denominator of the integrand by 16 and then making a u -substitution with $u = x/4$, we have

$$\begin{aligned} \int_{-M}^M \frac{4}{x^2 + 16} dx &= \frac{1}{4} \int_{-M}^M \frac{1}{(x/4)^2 + 1} dx \\ &= \tan^{-1}(x/4) \Big|_{-M}^M \\ &= \tan^{-1}(M/4) - \tan^{-1}(-M/4). \end{aligned}$$

Taking the limit as $M \nearrow \infty$, we find

$$\int_{-\infty}^{\infty} \frac{4}{x^2 + 16} dx = \pi/2 - (-\pi/2) = \pi.$$

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5. (15 points) (Bonus 9.2.19) Solve the IVP

$$\begin{cases} (x+1)y' - 2(x^2+x)y = \frac{e^{x^2}}{x+1} \\ y(0) = 5. \end{cases}$$

Solution: Dividing the equation by $x+1$, we get

$$y' - 2xy = \frac{e^{x^2}}{(x+1)^2}.$$

It follows that the integrating factor is $\mu = e^{-x^2}$. Thus,

$$\left[e^{-x^2} y \right]' = \frac{1}{(x+1)^2}.$$

Integrating both sides from 0 to x , we have

$$e^{-x^2} y - 5 = \int_0^x \frac{1}{(\xi+1)^2} d\xi = -\frac{1}{x+1} + 1.$$

Thus,

$$y = 6e^{x^2} - \frac{e^{x^2}}{x+1} = \frac{6x+5}{x+1} e^{x^2}.$$