

1. (8.2.8,8.2.72)

(a) (10 points) Find the indefinite integral

$$\int xe^{3x} dx.$$

(b) (15 points) Find the definite integral

$$\int_0^1 \tan^{-1} x dx.$$

Solution:(a) Integrate by parts with $u = x$ and $dv = e^{3x} dx$ so that $du = dx$ and $v = e^{3x}/3$, and

$$\int xe^{3x} dx = xe^{3x}/3 - \frac{1}{3} \int e^{3x} dx = xe^{3x}/3 - \frac{1}{9}e^{3x} + C.$$

(b) Integrate by parts with $u = \tan^{-1} x$ and $dv = dx$ so that $du = [1/(1+x^2)] dx$ and $v = x$, and

$$\int_0^1 \tan^{-1} x dx = x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2} \ln(1+x^2) \Big|_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

2. (8.3.46, 8.4.53)

(a) (10 points) Find the definite integral

$$\int_{-\pi/4}^{\pi/4} \tan^4 x \, dx.$$

(b) (15 points) Find the indefinite integral

$$\int \sqrt{9 - x^2} \, dx.$$

Solution:(a) Replace $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned} \int_{-\pi/4}^{\pi/4} [\tan^2 x \sec^2 x - \tan^2 x] \, dx &= \int_{-\pi/4}^{\pi/4} [\tan^2 x \sec^2 x - \tan^2 x] \, dx \\ &= \int_{-\pi/4}^{\pi/4} \tan^2 x \sec^2 x \, dx - \int_{-\pi/4}^{\pi/4} \tan^2 x \, dx \\ &= \int_{-1}^1 u^2 \, du - \int_{-\pi/4}^{\pi/4} [\sec^2 x - 1] \, dx \\ &= 2/3 - 2 + \pi/2 \\ &= \pi/2 - 4/3. \end{aligned}$$

We made a u -substitution $u = \tan x$.(b) Make the trig substitution $\sin \theta = x/3$ so $\cos \theta d\theta = (1/3) dx$ and $\sqrt{9 - x^2} = 3 \cos \theta$.

$$\begin{aligned} \int \sqrt{9 - x^2} \, dx &= \int 9 \cos^2 \theta \, d\theta \\ &= 9 \int \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= 9 \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right] + C \\ &= \frac{9}{4} [2 \sin^{-1}(x/3) + \sin(2 \sin^{-1}(x/3))] + C \\ &= \frac{9}{4} \left[2 \sin^{-1}(x/3) + 2 \frac{x \sqrt{9 - x^2}}{9} \right] + C \\ &= \frac{1}{2} [9 \sin^{-1}(x/3) + x \sqrt{9 - x^2}] + C. \end{aligned}$$

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3. (25 points) (8.5.21) Compute the indefinite integral

$$\int \frac{1}{(x+1)(x^2+1)} dx.$$

Solution: Use partial fractions:

$$\frac{a}{x+1} + \frac{bx+c}{x^2+1} = \frac{ax^2+a+bx^2+(b+c)x+c}{(x+1)(x^2+1)}.$$

$a+b=0=b+c$ and $a+c=1$. Therefore, $a=c=1/2$ and $b=-1/2$.

$$\begin{aligned}\int \frac{1}{(x+1)(x^2+1)} dx &= \frac{1}{2} \left[\int \frac{1}{x+1} dx - \int \frac{x-1}{x^2+1} dx \right] \\ &= \frac{1}{2} \left[\ln|x+1| - \frac{1}{2} \ln|x^2+1| + \tan^{-1}(x) \right] + C.\end{aligned}$$

4. (25 points) (8.8.28) Compute the definite integral

$$\int_0^1 \frac{4r}{1-r^4} dr.$$

Solution: There is a singularity in the integrand at $r = 1$ making this an improper integral requiring a limit to calculate. We also use the u -substitution $u = r^2$.

$$\begin{aligned}\int_0^1 \frac{4r}{1-r^4} dr &= \lim_{b \nearrow 1} \int_0^b \frac{4r}{1-r^4} dr \\ &= 2 \lim_{b \nearrow 1} \int_0^{b^2} \frac{1}{1-u^2} du.\end{aligned}$$

At this point, one can either proceed by partial fractions or by a trig substitution with $\sin \theta = u$. Either way, one finds

$$\begin{aligned}\int_0^1 \frac{4r}{1-r^4} dr &= 2 \lim_{b \nearrow 1} \int_0^{b^2} \frac{1}{1-u^2} du \\ &= \lim_{b \nearrow 1} \ln \left(\frac{1+b^2}{1-b^2} \right) \\ &= +\infty,\end{aligned}$$

because $(1+b^2)/(1-b^2)$ tends to $+\infty$ as $b \nearrow 1$.