1. (10 points) (5P30) Find the area of the bounded region in the first quadrant cut off by $3 = (\sqrt{x} + \sqrt{y})^2$.

Solution:

$$\begin{aligned} A &= \int_0^3 (\sqrt{3} - \sqrt{x})^2 \, dx \\ &= [3x - 2\sqrt{2}\left(\frac{2}{3}\right) x^{3/2} + x^2/2]\Big|_0^3 \\ &= 9 - 4\sqrt{6} + 9/2 \\ &= \frac{27 - 4\sqrt{6}}{2}. \end{aligned}$$

Name and section: ____________________

2. (10 points) (6.5.30) A pump rated at 1000 ft-lbs per second is used to pump 250 cubic feet of water (weighing 62.4 lbs per cubic foot) from a reservoir level $z = 0$ into a cylindrical tank with bottom 50 feet above the reservoir and radius 5 ft. Calculate how long this should take.

Solution: Notice that the tank will be filled to a height $H = 250/(25\pi) = 10/\pi$. The work required to pump the slice of water which ends up at height h in the tank is approximately

$$(50 + h)(62.4)(25\pi)\Delta h.$$

Thus, the total work required is

$$W = \int_0^{\pi/10} (50 + h)(62.4)(25\pi)\, dh = 25(62.4)\pi[5\pi + \pi^2/200].$$

This would be given in ft-lbs, and if we divide by the rating of the pump, we should obtain the total time in seconds:

$$\frac{62.4\pi^2(125 + \pi/8)}{1000} = \frac{15.6\pi^2(500 + \pi/2)}{1000} = \frac{7.8\pi^2(1000 + \pi)}{1000} = 7.8\pi^2(1 + \pi/1000],$$

or about 77 seconds, or a little over a minute.

Name and section: ____________________

3. (10 points) (7.1.28) Compute the definite integral

$$\int_{-2}^{0} 5^{-\theta}\, d\theta.$$

Solution: $u = -\theta \ln 5$.

$$\int_{-2}^{0} e^{-\theta \ln 5}\, d\theta = -\frac{1}{\ln 5}\int_{\ln 25}^{0} e^{u}\, du = \frac{1}{\ln 5} e^{u}\Big|_{0}^{\ln 25} = \frac{24}{\ln 5}.$$

Name and section: ____________________

4. (10 points) (8.4.50) Solve the initial value problem.

$$\begin{cases} \sqrt{x^2-9}\,y' = 1 & \text{for } 3 < x \\ y(5) = \ln 3. \end{cases}$$

Solution: Integrating $y' = 1/\sqrt{x^2-9}$, we obtain

$$\begin{aligned} y(x) - y(5) &= \int_5^x \frac{1}{\sqrt{t^2-9}}\,dt \\ &= \frac{1}{3}\int_5^x \frac{1}{\sqrt{(t/3)^2-1}}\,dt \\ &= \int_{5/3}^{x/3} \frac{1}{\sqrt{u^2-1}}\,du. \end{aligned}$$

At this point, we make a trigonometric substitution $\sec\theta = u$ according to which $\cot\theta = 1/\sqrt{u^2-1}$ and $du = \sec\theta\tan\theta\,d\theta$.

$$\begin{aligned} y(x) &= \ln(3) + \int_{5/3}^{x/3} \frac{1}{\sqrt{u^2-1}}\,du \\ &= \ln(3) + \int_{\sec^{-1}(5/3)}^{\sec^{-1}(x/3)} \sec\theta\,d\theta \\ &= \ln(3) + \ln(\sec\theta + \tan\theta)\Big|_{\sec^{-1}(5/3)}^{\sec^{-1}(x/3)} \\ &= \ln(3) + \ln(x/3 + \sqrt{x^2-9}/3) - \ln(3) \\ &= \ln\left(\frac{x+\sqrt{x^2-9}}{3}\right). \end{aligned}$$

Name and section: ____________________

5. (15 points) (9.2.3) Solve the first order linear equation

$$xy' + 3y = \frac{\sin x}{x^2} \quad \text{for } 0 < x.$$

Solution: Putting the equation into standard form, we get

$$y' + \frac{3}{x}y = \frac{\sin x}{x^3}.$$

Thus, the integrating factor should be

$$\mu = e^{\int (3/x)\,dx} = e^{3\ln x} = x^3.$$

Multiplying the equation by the integrating factor, we get

$$x^3y' + 3x^2y = (x^3y)' = \sin x.$$

Integrating, from some $x_0 > 0$ to x,

$$x^3y - x_0^3y(x_0) = -\cos x + \cos x_0.$$

Therefore,

$$y = \frac{1}{x^3}\left[-\cos x + \cos x_0 + x_0^3y(x_0)\right].$$

In this expression x_0 can be any positive constant, and $y(x_0)$ can be taken to be any constant. Alternatively, we can write simply

$$y = \frac{1}{x^3}\left[-\cos x + C\right]$$

where C is an arbitrary constant.

Name and section: ____________________

6. (15 points) (10P18) Discuss the convergence of the sequence $(-4)^n/n!$.

Solution: The limit is zero (and it helps to know this beforehand—see Theorem 5 in Chapter 10 Section 1). As long as $n > 5$, we have

$$\begin{aligned}
\left|\frac{(-4)^n}{n!}\right| &= \frac{4^n}{n!} \\
&< \frac{4^n}{5^{n-5}\,5!} \\
&= \frac{5^5}{5!}\left(\frac{4}{5}\right)^n \to 0.
\end{aligned}$$

Name and section: ____________________

7. (15 points) (10.7.27) Discuss the convergence of the series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}(x+2)^j}{j2^j}.$$

Solution: Denoting the j-th term by a_j, we attempt to use the ratio test:

$$\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} = \lim_{j\to\infty} \frac{|(x+2)|j2^j}{(j+1)2^{j+1}} = \frac{|x+2|}{2}.$$

Thus, the series converges absolutely for $|x+2| < 2$ and diverges for $|x+2| > 2$. When $x = -4$, the series becomes

$$\sum \frac{-1}{j}.$$

This is the negative harmonic series and diverges, for example, by the integral test. When $x = 0$, the series becomes

$$\sum -\frac{(-1)^j}{j}.$$

This is the (negative) alternating harmonic series and converges by the alternating series test.

Name and section: ____________

8. (15 points) (10.9.5) Find the Taylor series expansion for $\cos(5x^2)$ at $x = 0$ and discuss the convergence of the series.

Solution: We know

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}.$$

Therefore,

$$\cos(5x^2) = \sum_{j=0}^{\infty} \frac{(-25)^j}{(2j)!} x^{4j},$$

and this series converges (absolutely) for all x since the series for $\cos x$ converges for all x.