MAT 2401, Exam 1: (sample)

1. (25 points) (Ch 13 Review 4) Find the equation of the sphere with diameter the line segment with endpoints $(-1, 4, 2)$ and $(3, -2, 6)$.

Solution: The midpoint of the segment is the center: $((-1+3)/2, (4-2)/2, (2+6)/2)) = (1, 1, 4)$. The radius is half the length of the diameter:

$$\frac{\|(-1-3, 4+2, 2-6)\|}{2} = \frac{\sqrt{16+36+16}}{2} = \sqrt{17}.$$

The equation of the sphere is

$$\|(x, y, z) - (1, 1, 4)\| = \sqrt{17}$$

or

$$(x-1)^2 + (y-1)^2 + (z-4)^2 = 17.$$

2. (25 points) (Ch 13 Review 22) Find the unit vectors perpendicular to $3\vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$ and $-2\vec{\mathbf{i}} + 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$.

Solution: The cross product gives an orthogonal vector:

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 16 \end{pmatrix}$$

Thus, the two unit orthogonal vectors are $\pm(6, -1, 16)/\sqrt{293}$.

3. (25 points) (Ch 14 Review 11) Define a vector valued function of a real variable on the interval $[0, 2\pi]$ which traces out the ellipse $16x^2 + 4y^2 = 64$ once in the counterclockwise direction starting at $(0, 4)$.

Solution: First divide by 64 to put the ellipse in standard form:

$$\frac{x^2}{4} + \frac{y^2}{16} = 1.$$

This tells us that the horizontal semi-axis has length 2 and the vertical semi-axis has length 4. (The center is at the origin.) We can modify the vector valued function which traces a circle to get one that traces this ellipse.

Note that the formula $(\cos(t+\pi/2), \sin(t+\pi/2))$ gives a vector valued function which traces the unit circle in the counterclockwise direction starting from $(1, 0)$. It is easily checked that the vector valued function

$$\mathbf{r}(t) = (2\cos(t + \pi/2), 4\sin(t + \pi/2))$$

satisfies the equation of the ellipse. Thus, this gives the desired function. The expression can also be simplified using the angle addition formulas for sine and cosine:

$$\mathbf{r}(t) = (-2\sin t, 4\cos t).$$

4. (25 points) (Ch 14 Review 21) Find a vector valued function which traces the tangent line to the curve traced out by $\mathbf{r}(t) = (t^2+2t+1)\vec{\mathbf{i}}+(3t+1)\vec{\mathbf{j}}+(t^3+t+1)\vec{\mathbf{k}}$ at the point $(1,1,1)$ in the same direction as the motion determined by $\mathbf{r}$.

Solution: We first find the value of t corresponding to $(1,1,1)$. That would be $t=0$. (Use the middle coordinate, for example.)

Next, find the tangent vector $\mathbf{r}'(0)$. That would be $(2,3,1)$. The tangent line is therefore traced out by

$$\mathbf{f}(t) = (1,1,1) + t(2,3,1).$$