MAT 2401, Exam 2: (sample)

1. (25 points) (Ch 15 Review 6) A rectangular box is inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The sides of the box are parallel to the coordinate planes, and the vertices of the box are on the ellipsoid. Find a function of two real variables which gives the volume of the box. Explain clearly the construction of your function and indicate the domain and range.

Solution: We will take as the first variable the positive x-coordinate of a side of the box parallel to the y, z-plane. There is only one such side, and we will call the first variable α. Similarly, we will take as the second variable the positive y-coordinate of a side of the box parallel to the x, z-plane. Call this variable β. Note that $0 < \alpha < a$ and $0 < \beta < b$. Thus, the domain is

$$(0, a) \times (0, b) = \{(\alpha, \beta) : 0 < \alpha < a,\ 0 < \beta < b\}.$$

The volume is the product of the length (2α), width (2β), and height ($2c\sqrt{1 - (\alpha/a)^2 - (\alpha/b)^2}$). Thus, a formula for the volume function is

$$v(\alpha, \beta) = 8c\alpha\beta\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2}.$$

This function takes values in the interval $(0, \infty)$ and has range $(0, M]$ where M is the maximum possible value. What is this maximum possible value? Good question. To find M, we have to find places where the first partial derivatives vanish.

$$\frac{\partial v}{\partial \alpha} = 8c\beta\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2} + -\frac{8c\alpha^2\beta/a^2}{\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2}} = 8c\beta\frac{1 - 2\alpha^2/a^2 - \beta^2/b^2}{\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2}}.$$

$$\frac{\partial v}{\partial \beta} = 8c\alpha\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2} + -\frac{8c\alpha\beta^2/b^2}{\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2}} = 8c\alpha\frac{1 - \alpha^2/a^2 - 2\beta^2/b^2}{\sqrt{1 - \left(\frac{\alpha}{a}\right)^2 - \left(\frac{\beta}{b}\right)^2}}.$$

Setting each of these expression equal to zero, we must have

$$1 - 2\frac{\alpha^2}{a^2} - \frac{\beta^{2\,2}}{b} = 0 = 1 - \frac{\alpha^2}{a^2} - 2\frac{\beta^{2\,2}}{b}.$$

This means, in particular, that

$$\frac{\alpha^2}{a^2} = \frac{\beta^2}{b^2} = \frac{1}{3}.$$

So, for the max volume, $\alpha = a/\sqrt{3}$ and $\beta = b/\sqrt{3}$. That is,

$$M = v(a/\sqrt{3}, b/\sqrt{3}) = (8cab/3)\sqrt{1/3} = \frac{8abc\sqrt{3}}{9}.$$

2. (25 points) (Ch 15 Review 40) Calculate the first and second partial derivatives of $f(x, y) = x^2 e^{x/y}$.

Solution:

$$\frac{\partial f}{\partial x} = 2xe^{x/y} + x^2 e^{x/y}/y = \frac{xe^{x/y}}{y}(x + 2y).$$

$$\frac{\partial f}{\partial y} = -(x^3/y^2)e^{x/y}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x + 2y}{y}e^{x/y} + \frac{x(x + 2y)}{y^2}e^{x/y} = \frac{x^2 + 4xy + 2y^2)}{y^2}e^{x/y}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{3x^2}{y^2}e^{x/y} - \frac{x^3}{y^3}e^{x/y} = -\frac{x^2(3y + x)}{y^3}e^{x/y}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x^3}{y^3}e^{x/y} + \frac{x^4}{y^4}e^{x/y} = \frac{x^3(2y + x)}{y^4}e^{x/y}.$$

3. (25 points) (Ch 16 Review 11) Find the directional derivative of $f(x, y) = 3x^2 - 2xy^2 + 1$ at $(3, 2)$ toward the origin.

Solution: The direcitonal derivative in the direction $\mathbf{v}$, denoted $D_{\mathbf{v}} f$, is computed by taking a dot product of $\mathbf{v}$ with the gradient. Let's compute the gradient first:

$$\nabla f = (6x - 2y^2, -4xy).$$

$$\nabla f(3, 2) = (10, -24).$$

"In the direciton of the origin" means take $\mathbf{v}$ to be the unit vector pointing from $(3, 2)$ toward $(0, 0)$. That is,

$$\mathbf{v} = -(3, 2)/\sqrt{13}.$$

Finally, then

$$D_{\mathbf{v}} f(3, 2) = \nabla f(3, 2) \cdot \mathbf{v} = \frac{18}{\sqrt{13}}.$$

4. (25 points) (Ch 16 Review 45) Minimize $f(x, y) = x^2 + y^2 - 2x + 2y + 2$ on the closed disk $\{(x, y) : x^2 + y^2 \leq 4\}$. Indicate also points where local minima occur and the corresponding local minimum values.

Solution: We first look for places of vanishing gradient.

$$\nabla f = (2x - 2, 2y + 2).$$

Therefore, we are led to the point $(1, -1)$ which is certainly in the disk. The value of the function here is $f(1, -1) = 0$.

Next, we consider the boundary of the disk, which we parameterize as

$$\gamma(t) = (2\cos t, 2\sin t).$$

Plugging this into f, we get

$$g(t) = 4 - 4\cos t + 4\sin t + 2 = 6 - 4\cos t + 4\sin t.$$

We now use Calculus I techniques: Note that $g' = 4\sin t + 4\cos t$. This is zero when $\tan t = -1$. There are thus two critical points of interest on the circle corresponding to $t_1 = -\tan^{-1}(1) = -\pi/4$ and $t_2 = -\tan^{-1}(1) + \pi = 3\pi/4$.

It will be noted that $\cos t_1 = 1/\sqrt{2}$ and $\sin t_1 = -1/\sqrt{2}$. Thus, $f \circ \gamma(t_1) = g(t_1) = 6 - 8/\sqrt{2} = 6 - 4\sqrt{2} > 0$.

Similarly, $\cos t_2 = -1/\sqrt{2}$ and $\sin t_2 = 1/\sqrt{2}$, so $f \circ \gamma(t_2) = g(t_2) = 6 + 4\sqrt{2} > 0$.

It follows that the absoute minimum value is 0 which is achieved at $(1, -1)$.

One might ask if $\gamma(t_1) = (\sqrt{2}, -\sqrt{2})$ provides a local minimum, since t_1 gives a local minimum for g. To see that this is not the case, we can compute the directional derivative at $\gamma(t_1)$ in the direction toward the origin, that is the direction $-\gamma(t_1)/\|\gamma(t_1)\| = (-1/\sqrt{2}, 1/\sqrt{2})$.

Note that $\nabla f \circ \gamma(t_1) = (2\sqrt{2} - 2, -2\sqrt{2} + 2)$. In particular,

$$\nabla f \circ \gamma(t_1) \cdot \gamma(t_1)/\|\gamma(t_1)\| = -2 + \sqrt{2} - 2 + \sqrt{2} = -2(2 - \sqrt{2}) < 0.$$

This shows that the value of f decreases as one enters the disk from $\gamma(t_1)$.

Alternatively, one can observe that $f(x, y) = (x-1)^2 + (x+1)^2$ which is the standard form for a paraboloid with vertex at $(1, -1)$. From this, everything asserted above is clear.