MAT 2401 Final Exam: (sample)

1. (20 points) (Chapter 14 Review 43) Calculate the curvature of the image curve as a function of t if

$$\mathbf{r}(t) = (\cos(3t), -4t, \sin(3t)).$$

Solution:

$$\mathbf{r}' = (-3\sin(3t), -4, 3\cos(3t))$$
$$\gamma(s) = (\cos(3s/5), -4s/5, \sin(3s/5))$$
$$\ddot{\gamma}(s) = -(9/25)(\cos(3s/5), 0, \sin(3s/5))$$
$$\kappa = 9/25.$$

2. (20 points) (Chapter 15 Review 43) The plane $y = 2$ intersects the graph of $f(x, y) = 2x^2 + 3xy$ in a curve. Find a parameterization of the line tangent to the intersection curve at $(1, 2, 8)$.

Solution: We can parameterize the curve as a function of $t = x$:

$$\mathbf{r}(t) = (t, 2, 2t^2 + 6t).$$

We are interested in the tangent line at $t = 1$:

$$\ell(t) = (1, 2, 8) + t(1, 0, 10).$$

3. (20 points) (Chapter 16 Review 41) Classify all critical points of

$$f(x, y) = x^3 + y^3 - 18xy$$

Solution: We find the zeros of the gradient to determine the critical points.

$$Df = (3x^2 - 18y, 3y^2 - 18x) = 3(x^2 - 6y, y^2 - 6x)$$

Substituting $y = x^2/6$ into $3y^2 - 18x = 0$, we get $x^4/12 - 18x = 0$ or $x = 0$ or $x = 6$. These values lead to critical points at $(0, 0)$ and $(6, 6)$.

To determine the nature of the critical points, we compute the Hessian:

$$D^2 f = \begin{pmatrix} 6x & -6 \\ -6 & 6y \end{pmatrix}.$$

Thus, $\det D^2 f(0, 0) = -36 < 0$, so $(0, 0)$ is a saddle point. There are some directions in which the value of f increases and some in which it decreases. On the othe rhand, $\det D^2 f(6, 6) = 36^2 - 36 > 0$, so this is either a local min or a local max. In fact, $f_{xx}(6, 6) = 36 > 0$, so we have a local min at $(6, 6)$.

4. (20 points) (Chapter 17 Review 19) Evaluate

$$\int_V xyz\, dxdydz$$

Where V is the region boundef by $z = 2 - y^2$, $x = 0$, $y = 0$, and $y = x$.

Solution: The region is a wedge shaped region in the first octant. We can write the integral as an iterated integral as follows:

$$\begin{aligned}\int_0^{\sqrt{2}} \int_0^{y} \int_0^{2-y^2} xyz\, dzdxdy &= \int_0^{\sqrt{2}} \int_0^{y} xy(2-y^2)^2/2\, dxdy \\ &= \int_0^{\sqrt{2}} y^3(2-y^2)^2/4\, dy \\ &= \int_0^{\sqrt{2}} y^3(4-4y^2+y^4)/4\, dy \\ &= [4-(2/3)8+16/8]/4 = 1/6\end{aligned}$$

5. (20 points) (Chapter 18 Review 21) Let Γ be the boundary of the rectangle $[0,2] \times [0,1]$. Let $\mathbf{v} = (x - 2y^2, 2xy)$ on this curve and T its counterclockwise unit tangent vector. Calculate

$$\int_\Gamma \mathbf{v} \cdot T.$$

Solution: One can evalute this integral directly by paremeterizing the four sides of the rectangle and computing four line integrals, or one can use Green's theorem. I will use Gauss' (divergence) Theorem:

Let the integral be I, let R be the rectangular region bounded by Γ, and let n be the outward unit normal to R. Then rotating $\mathbf{v}$ and T clockwise by $\pi/2$, we get

$$\begin{aligned} I &= \int_\Gamma (2xy, 2y^2 - x) \cdot n \\ &= \int_R \operatorname{div}(2xy, 2y^2 - x) \\ &= \int_R (2y + 4y) \\ &= \int_0^2 \int_0^1 6y \, dy dx \\ &= 6. \end{aligned}$$