Calculus Review for ODE

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1 The Fundamental Theorem of Calculus

We begin with a mathematical (calculus) problem:

> *Find a function $y = y(x)$ whose derivative is given (prescribed) by another (continuous) function $f = f(x)$.*

This problem asks us to solve the equation $y'(x) = f(x)$ for the function y. The Fundamental Theorem of Calculus tells us how:

> *If $f = f(x)$ is any continuous function, x_0 is in the domain of definition of f, and y_0 is any constant, then the function*
>
> $$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$
>
> *has a well defined derivative, and that derivative is given by the formula*
>
> $$y'(x) = f(x).$$

The answer given by the Fundamental Theorem may be somewhat unexpected; there is not just one such function but a family of solutions. If we know specific choices for the starting point x_0 and the starting value y_0, then we get a unique solution.

Apply the Fundamental Theorem of Calculus to the following problems:

1. Find the function $y = y(x)$ whose derivative is given by $f(x) = x^2 + 3$ and which satisfies $y(2) = 1$.
2. Find the function $y = y(x)$ whose derivative is given by $f(x) = \sin^2 x$ and which satisfies $y(2) = 1$.
3. Find the function $y = y(x)$ whose derivative is given by $f(x) = x \sin^2 x$ and which satisfies $y(0) = 1$.
4. Find the function $y = y(x)$ whose derivative is given by $f(x) = \sin x^2$ which satisfies $y(0) = 1$.

Here is the same kind of problem reworded in a way that may be more familiar to you:

5. Find the velocity $v = v(t)$ of an object whose acceleration is given by $a(t) = -9.8$ (meters/second)/second and whose velocity at time $t = 2$ is 5 meters/second. Describe a physical system which might be described by this problem.

Here are some more:

6. A stone is dropped from a height of 98 meters. In how many seconds does it hit the ground?

7. An explosion causes debris to rise vertically with an initial velocity of 72 feet per second. In how many seconds does it attain a maximum height? What is the maximum height?

8. A stone was thrown up at a speed of 2 meters per second. After 2.5 seconds, the stone was caught by the person who threw it. What was the maximum height, and who threw the stone?

Finally, here are a couple "thinking" problems. (Everybody needs to be a thinker in this course.)

9. In the Fundamental Theorem why would it be wrong to say

$$y(x) = y_0 + \int_{x_0}^{x} f(x)\,dx\,?$$

10. The version of the Fundamental Theorem of Calculus with which you are probably familiar addresses how to compute an integral:

If $F = F(x)$ is a differentiable function whose derivative is $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Explain why these two versions really say the same thing. (Make sure you explain the correspondence between f, F, a, x, and b in this statement and y, f, x_0, x, y_0, and t in the one given at the beginning of this worksheet.)

2 Notes

1. The mathematical concepts on this worksheet should be familiar to you. If they are not, you may wish to review sections 5.1 and 5.2 of *Calculus* by Salas, Hille, and Etgen. Page 149 may also be of interest. If you want to read ahead, you can also review sections 18.1 and 7.6.

2. Problem 4.

   `http://www.math.unt.edu/integration_bee/AwfulTruth.html`

3. Problem 8.

   `http://hypertextbook.com/facts/2004/MichaelRobbins.shtml`