Math 2403
A Homework Problem—extra details

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Problem 2.6.4 in Brannon and Boyce concerns the initial value problem

$$\begin{cases} y' = 2t + e^{-ty} \\ y(0) = 1. \end{cases}$$

We discussed this problem in class, and I thought about it some more later.

Say we want to plot the slope field. We note that when t and y are large and positive, the exponential term should be small, and solutions (and the slope field) should be close to solutions of $y' = 2t$ which are quadratic functions $y = t^2 + c$. Thus, we know what the field looks like far away from the origin on the line $y = x$. In particular, solutions will be increasing when $t > 0$ in this region and decreasing when $t < 0$.

Similarly, when (t, y) is deep on the second and fourth quadrants, the exponential term will dominate and all solutions will be increasing.

The axes are also easy to check, since the exponential term is identically one on both axes. Thus, on the t-axis, solutions decrease for $t < 1/2$ and increase (with ever greater slope) for $t > 1/2$. The slope is 1 at the origin and for all points on the y-axis.

Beyond this, things are a little murky around the origin, especially to the left of the y-axis. (It's not hard to see, at least, that $y' > 1$ for all points in the first and fourth quadrants.)

To understand things a bit better, let's consider all the points for which $y' = 0$ in the slope field. This means

$$2t + e^{-ty} = 0$$

or, as we wrote down in class,

$$y = g(t) = \frac{1}{t} \ln \left(-\frac{1}{2t} \right).$$

We didn't do a thorough analysis of this function. (And we should have.) Obviously the graph of g is some kind of curve defined for $t < 0$. We did calculate the limit as $t \nearrow 0$ of $g(t)$ and we even got the correct answer, but we didn't know how to use it. Let's do it correctly now:

$$g'(t) = \frac{1}{t}(-2t)\left(\frac{1}{2t^2}\right) - \frac{1}{t^2}\ln\left(-\frac{1}{2t}\right) = -\frac{1}{t^2}\left[1 + \ln\left(-\frac{1}{2t}\right)\right].$$

Now, we are looking for zeros of $g'(t)$ for $t < 0$. It is easily seen that there is exactly one such zero at $t = -e/2$. Thus, the only critical point of g occurs at $(-e/2, g(-e/2)) = (-e/2, 2/e)$. Furthermore, it is not difficult to see that $g'(t) > 0$ for $t < -e/2$ and $g'(t) < 0$ for $t > -e/2$. It follows that there is exactly one well-defined curve with a single maximum on which the slope field is horizontal.

Now the limits come in:

$$\lim_{t \nearrow 0} g(t) = -\infty$$

by L'Hopital's rule, and

$$\lim_{t \searrow -\infty} g(t) = 0.$$

Thus, you can plot this function, and then it becomes clear how to interpret the slope field.

In particular, there is a critical solution $y_1(t)$ which increases from zero and passes through $(-e/2, 2/e)$ (with zero slope) and then increases to infinity becoming asymptotic to some quadratic $t^2 + c$. This is the long term asymptotic behavior for all solutions. And then there are three kinds of solutions that could possibly exist for all time besides this one:

1. Solutions above the critical $y_1(t)$ at $t = 0$ are always increasing.

2. Solutions below the critical solution but having positive values for some $t < 0$ will increase until they cross the curve $y = g(t)$. Then they will decrease until they cross that curve again for some t between $-e/2$ and 0. From there, they will increase to infinity.

3. Solutions which are always negative for $t < 0$ decrease from zero until they cross the graph of g at some t between $-e/2$ and 0. From there, they will increase to infinity.

In order to see this characterization involving the critical solution and the isocline which is the graph of g, it requires knowing and understanding the existence and uniqueness theorem for ODEs. That tells us, in particuar, since there is no singularity in the equation, that solutions cannot cross. It also tells us that the critical solution will exist for all time—well almost, since we already believe solutions heading into the first quadrant will become asymptotic to solutions of a linear ODE. Since the equation itself is nonlinear, it doesn't say anything about long time existence, especially where that exponential term becomes important. In particular, I would guess that there are also solutions which blow up to infinity in the second quadrant at some finite negative time. And I'll bet there are also solutions which "come from minus infinity" in the fourth quadrant at some finite positive time, and then enter the first quadrant. Obviously, the solution of our initial value problem can be neither of these, so that solution, I'll guess, exists for all time, and is one of the three kinds listed above. It's conceivable that it could be the critical solution, but that strikes me as exceedingly unlikely.

What is not really clear is which kind of solution is the one we get from the initial value problem. We can try to find out numerically. If the initial value were below $2/e$, then we would know we had one of the second or third type. However, $y(0) = 1 > 2/e$, so we really don't know. I'll post a mathematica notebook where I try to sort out the answer to this question numerically.