Math 2550, Exam 3 form a Name and section: ____________________

1. (25 points) (15.2.47) Evaluate the integral:

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx.$$

Hint: Change the order of integration.

Solution:

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx &= \int_0^\pi \int_0^y \frac{\sin y}{y}\, dxdy \\ &= \int_0^\pi \sin y\, dy \\ &= 2.\end{aligned}$$

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2. (25 points) (15.4.19) Evaluate the integral:

$$\int_0^{\ln 2} \int_0^{(\ln 2)^2 - y^2} e^{\sqrt{x^2+y^2}}\, dx dy.$$

Solution: Changing to polar coordinates:

$$\begin{aligned}
\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx dy &= \int_0^{\ln 2} \int_0^{\pi/2} e^r \, r \, d\theta dr \\
&= \frac{\pi}{2} \left[re^r \Big|_{r=0}^{\ln 2} - \int_0^{\ln 2} e^r \, dr \right] \\
&= \frac{\pi}{2} \left[2\ln 2 - (2-1) \right] \\
&= \frac{\pi}{2} \left(2\ln 2 - 1 \right).
\end{aligned}$$

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3. (25 points) (15.6.21) Let a, b, and c be fixed positive numbers, and let $\mathcal{V}$ be the volume

$$[0, a] \times [0, b] \times [0, c] = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\},$$

and let $\mathcal{V}$ model a solid of constant unit density $\delta \equiv 1$. Find the **moment of inertia** of this solid with respect to the z-axis.

Solution:

$$\begin{aligned} I &= \int_{\mathcal{V}} r^2 \\ &= \int_0^c \int_0^b \int_0^a (x^2 + y^2)\, dx dy dz \\ &= c \int_0^b \left(\frac{a^3}{3} + a y^2 \right) dy \\ &= \frac{c}{3}(a^3 b + a b^3) \\ &= \frac{abc}{3}(a^2 + b^2). \end{aligned}$$

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4. (25 points) (15.7.56) Find the volume of the solid

$$\mathcal{V} = \left\{ (x,y,z) \in \mathbb{R}^3 : 1 \le \sqrt{x^2+y^2+z^2} \le 1 + \frac{z}{\sqrt{x^2+y^2+z^2}},\ 0 \le z \le 2 \right\}.$$

Hint: $z/\sqrt{x^2+y^2+z^2} = \cos\phi$ where ϕ is the azimuthal angle.

Solution: The volume is given by

$$\begin{aligned}
\int_{\mathcal{V}} 1 &= \int_0^{\pi/2} \int_0^{2\pi} \int_1^{1+\cos\phi} \rho^2 \sin\phi \, d\rho d\theta d\phi \\
&= 2\pi \int_0^{\pi/2} \frac{1}{3}[(1+\cos\phi)^3 - 1] \sin\phi \, d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi/2} (3\cos\phi + 3\cos^2\phi + \cos^3\phi) \sin\phi \, d\phi \\
&= -\frac{2\pi}{3} \int_1^0 (3u + 3u^2 + u^3) \, du \\
&= \frac{2\pi}{3} \left(\frac{3}{2} + 1 + \frac{1}{4} \right) \\
&= \frac{2\pi}{3} \left(\frac{11}{4} \right) \\
&= \frac{11\pi}{6}.
\end{aligned}$$