The shape and equation of a hyperbola

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Here we consider the relation

$$y^2 - x^2 = 1.$$

That is, we consider the set of points (x, y) for which $y^2 - x^2 = 1$. Calling this set $\mathcal{R}$ and writing it in mathematical notation, we wish to understand

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : y^2 - x^2 = 1\}.$$

Note that, for example, $(x, y) = (0, 0) \notin \mathcal{R}$ but $(x, y) = (0, 1) \in \mathcal{R}$.

Exercise 1 *Find a point $(\sqrt{2}, a)$ in the first quadrant which is also in $\mathcal{R}$. Can you find a point $(a, a) \in \mathcal{R}$?*

It is well known that the full set of points in the plane determined by this relation is a curve (or pair of curves) called a **hyperbola**, but let's imagine we don't know that and see what we can find out on our own.

For points where $y > 0$, we have $y = \sqrt{x^2 + 1}$. Interpreting the expression on the right as a form of the Pythagorean theorem, or the distance formula, we can write it as

$$\sqrt{(x - 0)^2 + (1 - 0)^2},$$

and if we also have $x > 0$, then we see we are looking for points in the first quadrant with the following property:

> *For each $x > 0$, the distance from the point $(x, 1)$ to the origin $(0, 0)$ is the same as the distance (that is, the height $y > 0$) from $(x, 0)$ to (x, y).*

It is intuitively clear from this description that

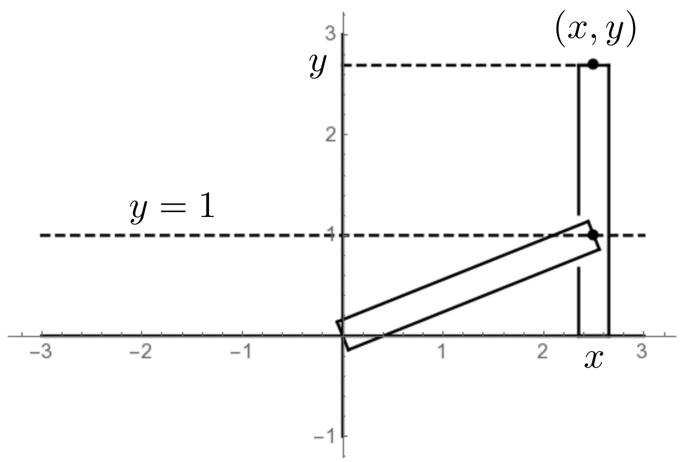

Figure 1: a point on a hyperbola

1. For each $x > 0$, there is a unique $y > 0$ and the dependence of y on x is continuous. In other words: The intersection of $\mathcal{R}$ with the first quadrant is a (continuous) curve.

2. As x tends to zero (from the right) the value of y tends to $y = 1$. The point $(0, 1)$ is a terminal point on the curve.

3. The curve given by the graph of $y = \sqrt{x^2 + 1}$ is increasing and tends to infinity with x.

We could rigorously justify these assertions by applying calculus techniques to $y = \sqrt{x^2 + 1}$, and we could also, for example, extend the curve symmetrically as the graph of the same even function into the second quadrant and prove that this curve is convex. Let us consider something in this direction. It is well-known that a hyperbola is **asymptotic** to straight lines. Let's see if we can verify this. If we assume there is an asymptotic line of the form $y = mx$, then we need to choose m so that

$$\lim_{x \to \infty} \sqrt{x^2 + 1} - mx = 0. \tag{1}$$

This is what it means for the curve to be asymptotic to the line (in the first quadrant). Applying some algebra to the difference, we get

$$\sqrt{x^2 + 1} - mx = (\sqrt{x^2 + 1} - mx)\,\frac{\sqrt{x^2 + 1} + mx}{\sqrt{x^2 + 1} + mx} = \frac{x^2 + 1 - m^2 x^2}{\sqrt{x^2 + 1} + mx}.$$

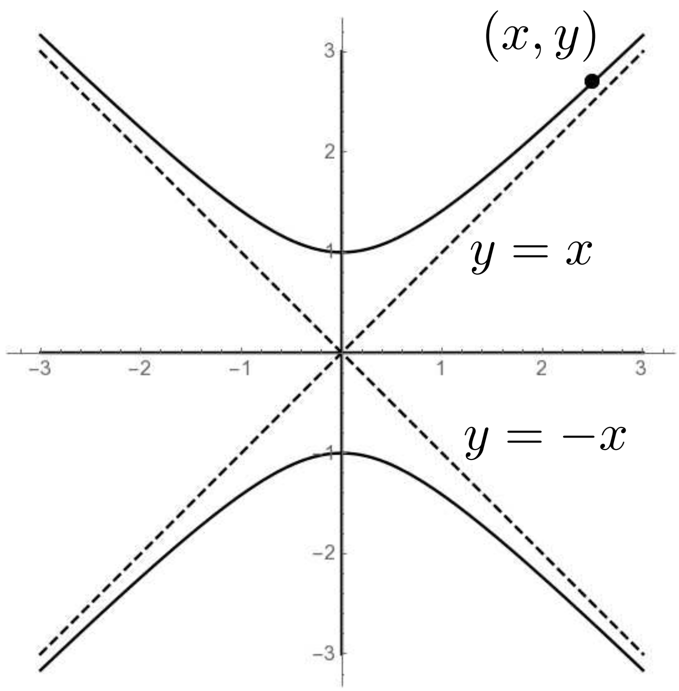

Figure 2: all points on a hyperbola

The numerator in the last expression is $(1-m^2)x^2+1$, and it is clear that if we choose $m=1$, we get (1).

Exercise 2 *Let $\mathcal{R}_1$ denote the portion of $\mathcal{R}$ in the first quadrant. The intersections of $\mathcal{R}$ with the other quadrants are symmetric to $\mathcal{R}_1$ and are given by*

$$\begin{aligned}\mathcal{R}_2 &= \{(-x,y) : (x,y) \in \mathcal{R}_1\},\\ \mathcal{R}_3 &= \{(-x,-y) : (x,y) \in \mathcal{R}_1\}, \text{ and}\\ \mathcal{R}_4 &= \{(x,-y) : (x,y) \in \mathcal{R}_1\}.\end{aligned}$$

At this point we know the rough shape of $\mathcal{R}$ is that shown in Figure 2. Finally, we consider another known property of hyperbolas which gives a method of construction alternative to the one we have introduced in association with Figure 1. For a hyperbola like ours with **center** $(0,0)$, there should be two **focal points** $(0,q)$ and $(0,-q)$ determined by some number $q>1$, and $\mathcal{R}$ should be the set of points (x,y) such that the **difference** of the distances from (x,y) to the focal points is a constant.

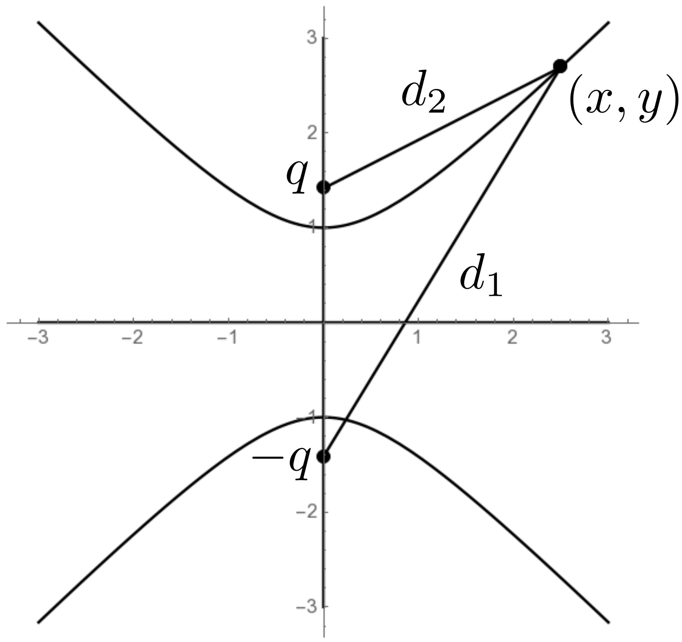

Figure 3: $d_1 - d_2 = c$

We may also recall, at this point, that an ellipse is generated by keeping the **sum** of the distances to two focal points constant. Technically, if we order the focal points, the upper branch of the hyperbola and the lower branch will give differences of the opposite sign (but the absolute values of the differences of the distances to the focal points will be constant). Taking $(x, y) \in \mathcal{R}_1$, the description we have given for the hyperbola translates into the equation

$$\sqrt{x^2 + (y + q)^2} - \sqrt{x^2 + (y - q)^2} = c \qquad (2)$$

for some postive constant c.

We ask the following questions:

1. Is this correct, i.e., can we choose $q > 1$ and $c > 0$ so that the relation (2) reduces to $y^2 - x^2 = 1$?

2. If so, what are the correct choices of q and c?

Again, we apply some algebra. Rearranging and squaring the relation in (2), we find

$$x^2 + (y + q)^2 = c^2 - 2c\sqrt{x^2 + (y - q)^2} + x^2 + (y - q)^2.$$

This simplifies to

$$4qy - c^2 = -2c - 2c\sqrt{x^2 + (y-q)^2}.$$

Squaring again,

$$16q^2y^2 - 8qc^2y + c^4 = 4c^2(x^2 + y^2 - 2qy + q^2).$$

Simplifying, this becomes

$$(16q^2 - 4c^2)y^2 - 4c^2x^2 = 4q^2c^2 - c^4. \tag{3}$$

This looks like it is starting to have the correct form. At this point, however, we need to be a little careful. We might think we can choose q and c so that all the coefficients are 1. That is,

$$\begin{aligned} 16q^2 - 4c^2 &= 1, \\ 4c^2 &= 1, \qquad \text{and} \\ 4q^2c^2 - c^4 &= 1. \end{aligned}$$

This would mean (by the middle equation) that $c = 1/2$.

Exercise 3 *What happens if we take $c = 1/2$ and try to choose q so that the other two coefficients are 1?*

We shouldn't be so ambitious. What we should do, is try to choose q and c so that all the coefficients are **the same**. This gives the weaker condition

$$16q^2 - 4c^2 = 4c^2 = 4q^2c^2 - c^4.$$

The first equation gives $c^2 = 2q^2$. Substituting this in the equation on the right (after canceling a factor of c^2) we get

$$4q^2 - 2q^2 = 4 \qquad \text{or} \qquad q = \sqrt{2}.$$

Notice that this gives a value of $q > 1$ and leads to $c = 2$. With these (unique) choices the relation (2) reduces to the expression $y^2 - x^2 = 1$ with which we started. We also see the difference of the distances to the focal points is $c = 2$. For points in $\mathcal{R}_3 \cup \mathcal{R}_4$ we get

$$\sqrt{x^2 + (y+q)^2} - \sqrt{x^2 + (y-q)^2} = -2.$$

Exercise 4 *Can you give an insightful explanation for why it is impossible to choose q and c so that the coefficients in (3) all become 1?*