An improper integral (from the homework)

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\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+4)(y^2+4)}\,dxdy &= \lim_{A,B\nearrow\infty}\int_{-B}^{B}\int_{-A}^{A}\frac{1}{(x^2+4)(y^2+4)}\,dxdy\\
&= \lim_{A,B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}\int_{-A}^{A}\frac{1}{x^2+4}\,dxdy\\
&= \frac{1}{4}\lim_{A,B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}\int_{-A}^{A}\frac{1}{(x/2)^2+1}\,dxdy\\
&= \frac{1}{2}\lim_{A,B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}\int_{-A}^{A}\frac{1}{u^2+1}\,dudy\\
&= \frac{1}{2}\lim_{A,B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}[\tan^{-1}(A)-\tan^{-1}(-A)]\,dy\\
&= \frac{1}{2}\lim_{A,B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}[\tan^{-1}(A)+\tan^{-1}(A)]\,dy\\
&= \lim_{A,B\nearrow\infty}\tan^{-1}(A)\int_{-B}^{B}\frac{1}{y^2+4}\,dy\\
&= \frac{\pi}{2}\lim_{B\nearrow\infty}\int_{-B}^{B}\frac{1}{y^2+4}\,dy\\
&= \frac{\pi^2}{4}.
\end{aligned}
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