Riemann Integration on Objects

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In the second semester of calculus (Calc II) one learns about the Riemann integral on intervals:

$$\int_a^b f(x)\,dx.$$

The construction used to define the Riemann integral on intervals may be generalized to apply to integration on a variety of sets, which we are calling "objects." It is useful to have an idea of the general abstract construction of such integrals so that one can apply it to objects such as

1. regions (areas) in $\mathbb{R}^2$,
2. volumes in $\mathbb{R}^3$,
3. regions in $\mathbb{R}^n$ for $n > 3$ (i.e., "hypervolumes" or n-dimensional manifolds),
4. curves in $\mathbb{R}^n$,
5. surfaces in $\mathbb{R}^n$.

Integration on all these "objects" is possible, and we describe such integration here in abstract terms.

1 Abstract Integration

Several ingredients are required. Most are very easy to understand. We need an object, which we will call, generally, $\mathcal{V}$ and a real valued function defined on the object:

$$f : \mathcal{V} \to \mathbb{R}.$$

The function f assigns a real number to each point in $\mathcal{V}$. The basic concept we want to define/discuss is called

Integrating a real valued function f on an object $\mathcal{V}$.

Partition

Next we need a way to cut the object $\mathcal{V}$ up into small pieces. The set of pieces together is called a **partition** and is sometimes denoted by

$$\mathcal{P} \qquad \text{or} \qquad \{\mathcal{V}_j\}.$$

There should be finitely many pieces in the partition, and they are indexed by j. Thus, $\{\mathcal{V}_j\}$ means $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_k\}$ where k is some finite number. An important thing to remember about a partition is that the **union** of the partition pieces gives the entire object. In mathematical notation this is expressed by writing

$$\mathcal{V} = \cup_j \mathcal{V}_j.$$

As a technical point, it's often okay for the partition pieces to overlap a little bit, but you don't want them to overlap too much. We will mention this technical point again below.

Measure

We also need a way to measure the size of the partition pieces. Technically, we often need *two ways* to measure partition pieces, but we'll get to that in a moment. The first way, we'll call a **measure** and denote the size of a piece by

$$\text{meas}(\mathcal{V}_j).$$

If $\mathcal{V} = [a, b]$ is an interval, then we can partition into subintervals $\mathcal{V}_j = [x_{j-1}, x_j]$ where $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_k = b$, and we can use length for the measure: $\text{meas}(\mathcal{V}_j) = x_j - x_{j-1}$. If $\mathcal{V}$ is an area in $\mathbb{R}^2$, then we will want $\text{meas}(\mathcal{V}_j) = \text{area}(\mathcal{V}_j)$. You'll note that area and length are very different kinds of measures. We need to choose the measure suitable to the object on which we want to integrate. The measure tells us how much the partiation pieces may overlap: We always want $\text{meas}(\mathcal{V}_i \cap \mathcal{V}_j) = 0$ when $i \neq j$. You can check that this happens with our partition of the interval.

Riemann Sum

Once we have a partition and a measure, we can form a **Riemann sum**:

$$\sum_j f(\mathbf{x}_j^*)\ \mathrm{meas}(\mathcal{V}_j).$$

Actually, we also needed to choose the **evaluation points** $\mathbf{x}_j^*$, but that almost goes without saying. These points are chosen so that $\mathbf{x}_j^* \in \mathcal{V}_j$. There are k of them, and the superscript "$*$" indicates that there might be many alternative choices, i.e., there are probably many points in $\mathcal{V}_j$ from which to choose. Eventually, we want to consider **all possible** choices, but for now we can just imagine having made one such choice.

It will be recalled that the Riemann sum associated with a function defined on an interval $[a, b]$ can be interpreted as an approximation of the (signed) area under the graph of the function. There are various similar interpretations which are, more or less, possible for other Riemann sums.

The Riemann Integral

We are essentially ready to define the integral of f over $\mathcal{V}$. Let us write down the definition and explain it.

$$\int_{\mathcal{V}} f = \lim_{\|\mathcal{P}\|\to 0} \sum_j f(\mathbf{x}_j^*)\ \mathrm{meas}(\mathcal{V}_j). \tag{1}$$

This says that we get the integral by taking a limit of the Riemann sum(s). How many Riemann sums are there? There are usually a lot of them, because there are, first of all, many choices for the partition. Also, there can be many choices for the evaluation points. Having noted and understood all these possible choices, it's relatively easy to explain what is going on in this limit—though this is probably the trickiest part.

Norm of a Partition

Before we give that explanation, we need one more thing. We need to explain $\|\mathcal{P}\|$ which is the **norm of the partition**. The condition $\|\mathcal{P}\| \to 0$ is a (second) way of saying we want all the partiation pieces to get "small." For intervals, we can use the measure to accomplish this:

$$\|\mathcal{P}\| = \max_j (x_j - x_{j-1}).$$

That is, the norm of a partition consisting of intervals is the length of the longest interval in the partition. If we try to use area, on the other hand, for the norm of a partition consisting of areas in the plane, then we run into trouble. Can you see why? The problem is that a very small area in the plane (arbitrarily small actually) can be spread out all over the place.

Exercise 1 *Find a subset $\mathcal{V}_1$ of the unit square $\mathcal{V} = [0,1] \times [0,1]$ with the following properties:*

1. $\text{area}(\mathcal{V}_1) = 1/1000$*, and*
2. *Given* **any** *point $p \in \mathcal{V}$, there is a point $q \in \mathcal{V}_1$ such that $|p - q| < 1/1000$.*

To elaborate on this problem a little more, we'd like (at least when the function f is continuous) to have the individual terms $f(\mathbf{x}_j^*)\,\text{meas}(\mathcal{V}_j)$ in the Riemann sum to have about the same value no matter which evaluation point $\mathbf{x}_j^* \in \mathcal{V}_j$ is chosen. This amounts to having $f(\mathbf{x}_j^*)$ and $f(\mathbf{x}_j^{**})$ close together whenever $\mathbf{x}_j^*$ and $\mathbf{x}_j^{**}$ are any two different evaluation points in $\mathcal{V}_j$. This kind of situation is most easily realized by requiring $\mathcal{V}_j$ does not "spread out" too much. And the measure area does not accomplish this.

In short, we often need another way to measure the size of pieces. Without specifying this measure exactly, let us denote it by diam. Then we can take (at least symbolically)

$$\|\mathcal{P}\| = \max_j \text{diam}(\mathcal{V}_j).$$

Thus, $\|\mathcal{P}\|$ is the "diameter" of the partition piece with the largest "diameter." For many kinds of sets,

$$\text{diam}(\mathcal{V}_j) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in \mathcal{V}_j\} \tag{2}$$

provides a reasonable notion of diameter. In this expression, the sup (supremum) of a set of real numbers is the *smallest number that is greater than every number in the set.* That's kind of a mouthful. The supremum is also sometimes called the *least upper bound*, and it's an axiom of the real numbers that any bounded set always has a well-defined supremum or least upper bound. This is basically the way mathematicians try to avoid having gaps in the real number line. For this reason, the axiom requiring the existence of least upper bounds for bounded sets of real numbers is called the **axiom of completeness** as well as the **least upper bound axiom**. But this is taking us on a bit of a tangent. The points are that (1) the existence of the supremum in the definiton of diameter (2) relies on a mathematical axiom about the real numbers—it's not something you can prove and it's something rather complicated—and (2) it is generally possible to make a definition of diameter that makes sense.

The Limit

The existence of the limit in (1) means the following thing:

There is a number L such that for any (small) positive real number $\epsilon > 0$, there is another positive real number $\delta > 0$ having the property that whenever $\|\mathcal{P}\| < \delta$, then

$$\left| L - \sum_j f(\mathbf{x}_j^*) \text{ meas}(\mathcal{V}_j) \right| < \epsilon.$$

That is, if the largest partition piece has size/diameter smaller than δ, then (no matter how you pick the partition subject to the size requirement and no matter how you pick the evaluation points) then the Riemann sum will be "ϵ close" to the limit L.

If there is such a number L, then we call that number the integral:

$$L = \int_{\mathcal{V}} f.$$

Naturally, this integral on a general object may have various interpretations as the area, volume, or hyper-volume "under" the graph of the function f or some other geometric quantity.

If we were to desire more technical precision, it would be useful to give conditions on the sets $\mathcal{V}$, the partitions $\{\mathcal{V}_j\}$ and the functions $f : \mathcal{V} \to \mathbb{R}$ for which the limit (i.e., the limit of the Riemann sums) actually exists, so that the integral exists and is well-defined. One condition is well-known to imply the existence of Riemann integrals in many cases, so we mention it as a vaguely stated theorem.

Theorem 1 *If $\mathcal{V}$ is a (closed) set on which it makes sense for a real valued function $f : \mathcal{V} \to \mathbb{R}$ to be* **continuous***, and the function f is indeed* **continuous***, then the Riemann integral*

$$\int_{\mathcal{V}} f$$

is well-defined.

A somewhat more precise statement is given on page 896 of the Thomas Calculus text; see also Theorem 1 in section 5.3, page 319.

2 A Difference

The integration we have introduced, even in the case where $\mathcal{V} = [a, b]$ is an interval, is not exactly the same as the integration on integrals from Calc II. Also, there

are various notions of integration on curves and surfaces in calculus that are usually introduced with an analogous difference. To emphasize this difference, the integration on objects we have introduced is sometimes called **integration on sets** as opposed to integration on **oriented sets**. To see how orientation plays a role, let's restrict to intervals. Given an interval $[a, b]$ it's true that

$$\int_a^b f(x)\,dx = \int_{[a,b]} f.$$

These two kinds of integrals are **exactly** the same. However, using the notion of integration from Calc II, it also makes perfectly good sense to write

$$\int_b^a f(x)\,dx \tag{3}$$

and integrate "backwards" on the interval $[a, b]$. You will recall that there is a host of manipulations associated with this kind of backwards integration. For example, we know

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Not only is the backwards integral in (3) difficult to express as an integral on a set, but these two kinds of integrals have different change of variables formulas. Let's start with a change of variables in Calc II, or what was called "u-substitution." Quite generally, if we have a change of variables $u = \psi(x)$, then in Calc II we would write

$$\int_a^b f(x)\,dx = \int_{\psi(a)}^{\psi(b)} \frac{f \circ \psi^{-1}(u)}{\psi' \circ \psi^{-1}(u)}\,du. \tag{4}$$

You may not remember the formula looking this complicated. What you may remember is more along the following lines: From $u = \psi(x)$, you have $du = \psi'(x)\,dx$, so

$$\int_a^b g(x)\psi'(x)\,dx = \int_{\psi(a)}^{\psi(b)} g \circ \psi^{-1}(u)\,du. \tag{5}$$

If you compare, you will see that these are saying the same thing, but there is a preconditioning of the integrand $f(x)$ obtained by writing

$$g(x) = \frac{f(x)}{\psi'(x)}.$$

Our point here, however, is that this formula works even when $\psi(a) < \psi(b)$ so that the integral you get when you change variables is a backwards integral.

The corresponding computation for integration on sets is usually expressed in terms of a **scaling factor** σ, and when integrating on (unoriented) sets, the scaling factor is always the **absolute value** of something. In the simple case of a change of variables $\psi : [a, b] \to \mathbb{R}$, the scaling factor is

$$\sigma = |(\psi^{-1})'(u)| = \frac{1}{|\psi' \circ \psi^{-1}(u)|}. \tag{6}$$

Exercise 2 *Differentiate the relation*

$$\psi \circ \psi^{-1}(u) = u$$

to obtain the equivalent expressions for the scaling factor σ in (6). Hint: Be careful with your differentiation and the use of the chain rule.

When $\psi(a) < \psi(b)$, then we can use set integration to express either of the u-substitution rules (4) or (5). For example,

$$\int_{[a,b]} f = \int_{[\psi(a),\psi(b)]} f \circ \psi^{-1}(u)\, \sigma(u).$$

When ψ reverses the direction of the interval and $\psi(b) < \psi(a)$, then the change of variables formula is, superficially, a little different:

$$\int_{[a,b]} f = \int_{[\psi(b),\psi(a)]} f \circ \psi^{-1}(u)\, \sigma(u).$$

This is where the absolute value comes in with regard to the scaling factor, since in this case $\sigma = -1/\psi' \circ \psi^{-1}(u)$.

Warning: You may be used to using the "scaling factor"

$$\frac{1}{\psi' \circ \psi^{-1}(u)}\, du$$

in formula (4) when changing variables in the integral on the oriented interval $[a, b]$, but for integration on an interval as a set, the correct scaling factor is

$$\sigma = \left| \frac{1}{\psi' \circ \psi^{-1}(u)} \right|.$$

3 Calculation/Concrete Integration in one special case

You may leave the discussion above with a feeling of dissatisfaction. In fact, it can be very useful to understand the abstract notion of "integration on objects" we have described above. There are problems that are important for engineers and other people who use mathematics which are very difficult to work without it. On the other hand, if one wants to compute a concrete answer, then the definition

$$\int_{\mathcal{V}} f = \lim_{\|\mathcal{P}\|\to 0} \sum_j f(\mathbf{x}_j^*)\,\mathrm{meas}(\mathcal{V}_j)$$

is of limited use. The same was true in Calc II, and that's why the preponderance of the course was spent on various "techniques" of integration. First you may have used the Riemann sum definition to derive some simple examples like

$$\int_{[a,b]} x^n = \frac{1}{n+1}(b^{n+1} - a^{n+1}),$$

or you may have just memorized the "power rule." At any rate, you eventually memorized some elementary integration formulas for powers, trigonometric functions, and exponential functions. Then you learned other techniques, like u-substitution, various algebraic tricks and partial fractions for rational functions, integration by parts, and others. The definition using Riemann sums was probably lost in the shuffle, but that definition is really what gives meaning to the rest.

The good news is that, for the most part, there is no correspondent torrent of complicated techniques needed to integrate on sets that are more complicated than intervals. Usually, one reduces the calculation of an integral on any set to some equivalent calcuation involving 1-D integrals on intervals—which are then attacked with the techniques from Calc II. As one example of this, let us consider integration on a three-dimensional volume in a couple simple cases.

Integration on a Rectangular Parallelopiped, i.e., A Box

If

$$\mathcal{V} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : a_j \le x_j \le b_j,\ j = 1, 2, 3\}$$

is a Cartesian product of three intervals and $f : \mathcal{V} \to \mathbb{R}$ is a continuous function defined on $\mathcal{V}$, then

$$\int_{\mathcal{V}} f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3)\, dx_3 dx_2 dx_1.$$

The expression on the right is what's called an **iterated integral**. It just means

First evaluate

$$\int_{a_3}^{b_3} f(x_1, x_2, x_3)\, dx_3$$

as a 1-D integral, thinking of x_1 and x_2 as constants. Then take the result, which will be a function $g = g(x_1, x_2)$, and evalute the 1-D integral

$$\int_{a_2}^{b_2} g(x_1, x_2)\, dx_2 = \int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x_1, x_2, x_3)\, dx_3 \right) dx_2$$

thinking of x_1 as a constant. Finally take the result, which will be a function $h = h(x_1)$, and evalute the 1-D integral

$$\int_{a_1}^{b_1} h(x_1)\, dx_1 = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x_1, x_2, x_3)\, dx_3 \right) dx_2 \right) dx_1.$$

The fact that this technique of "iterated integrals" works is called **Fubini's Theorem**.

A Generalization

A version of Fubini's theorem works for a somewhat more general region $\mathcal{V} \subset \mathbb{R}^3$ having the form

$$\mathcal{V} = \{(x, y, z) : \psi_1(x, y) < z < \psi_2(x, y),\ (x, y) \in \mathcal{U}\}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is a region in the plane and ψ_1 and ψ_2 are real valued functions with domain $\mathcal{U}$ satisfying $\psi_1 < \psi_2$. In this case, it is possible to express the integral of a function f over $\mathcal{V}$ as the iterated integral

$$\int_{\mathcal{V}} f = \int_{\mathcal{U}} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right).$$

This reduction of triple/volume integrals to iterated integrals is discussed in the Thomas text on pages 922-924. Integration on various regions $\mathcal{U} \subset \mathbb{R}^2$ is found in sections 15.2-15.4.

In any case, for our purposes, the main points are:

1. It's useful to know the theory of Riemann sums for integrals on many kinds of objects.

2. There are ways to reduce many integrals on various objects to iterated 1-D integrals.