# Calculus I and II
# A Persective for Multivariable Calculus

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## Calculus so far...

I am teaching a course called *Introduction to Multivariable Calculus* this semester. After looking over the course outline, I had some thoughts both on this particular course and on the perspective that a student, having "finished" Calculus I and Calculus II (and maybe some linear algebra) might have at the outset of learning Calculus III Multivariable Calculus. I suspect that my thoughts on these matters will be of interest to some minority of my students, but not all. I suspect, furthermore, that these thoughts of mine will be really useful to an even smaller number of my students. With these suspicions in mind, and viewing the material that is, in fact, to be covered, I decided that I would not spend the first lecture nor part of the first lecture addressing these ideas. Nevertheless, I would try to prepare a document (this document) and the students can make of it what they can or will.

To simplify matters a little bit, Calculus I and Calculus II treat four main questions:

1. What is a function (especially a *real valued function of one real variable*)?
2. What is differentiation?
3. What is integration?
4. What is the relationship between integration and differentiation?

It would be nice to imagine that all students starting Multivariable Calculus have understood the answers to these questions, or at least most facets of them, in significant depth and detail. In reality, most students understand some fairly superficial aspects of calculational methods involving differentiation and integration. Few students have a working understanding of Euler's deep notion of a function. Few students really get the notion of a difference quotient or a Riemann sum, and for most, the fundamental theorem of calculus is little more than a method to evaluate definite integrals.

For many of you students, therefore, this is a good time to broaden your perspective on the details of Calculus I and II and deepen your understanding of the answers to the above questions. Let's see what I can do to help you to that end.

# Functions

Calculus I and Calculus II are courses about a *real valued function of one real variable*.

> *What is a* **real valued function of one real variable***?*

In order to understand the answer, let's start with the more general notion of a function.

Given two sets $X$ and $Y$, a **function** is a rule which assigns to each $x$ in the set $X$ a unique $y$ in the set $Y$.

The set $X$ is called the **domain** of the function.

In the definition I have given, I did not give the rule a name. But we can give the rule a name. For example, let's call it $f$. Then we write

$$f : X \to Y \qquad \text{and} \qquad y = f(x).$$

These suggestive notations can be read "$f$ is a function from $X$ to $Y$" and "the value assigned by $f$ to the domain element $x \in X$ is $y \in Y$." The symbol "$\in$" in "$x \in X$" and "$y \in Y$" means "$x$ is an element of the set $X$" and "$y$ is an element of the set $Y$." In the sentence above, the reading is a little more clear if one puts in "which":

> The value assigned by the rule $f$ to the element $x$ which is an element of the set $X$ is $y$ which is an element of the set $Y$.

This notion of a function was introduced by the Mathematician Leonhard Euler, and though it seems simple, this idea of inputs and outputs took a long time to understand clearly.

In Calculus I and Calculus II the sets $X$ and $Y$ are subsets of the set of real numbers $\mathbb{R}$. In particular, we can basically assume $X = (a, b)$ is an open interval in the real line, so that $x \in X$ means $a < x < b$, and we can assume $Y = \mathbb{R}$. This is what makes a function a "real valued function of one real variable." The **values** of the function are elements of $Y$, so that's why we say $f$ is **real valued** when $Y = \mathbb{R}$. When one talks about a "variable" here, one means the elements of the domain. The phrase "one real variable" means roughly "real numbers on a specified interval": $X = (a, b)$. It will be clearer after you learn "multivariable calculus" why the emphasis is on "one" in this phrase.

Now, let's connect this to something you probably remember and clarify what that was about. When someone says to you "Consider the function $f(x) = x^2$," what they want you to do is think about the rule that assigns the number $y = x^2$ to the variable element $x$. Let's say, we make a further restriction that the domain of our function $f$ is the open interval $(-1, 1)$. What probably comes to your mind is the picture in Figure 1. This is not a bad picture to have in mind, but it may not be

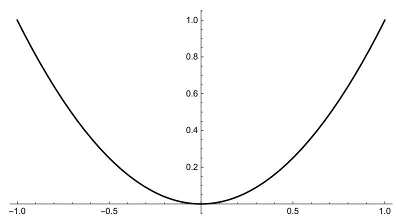


Figure 1: the "function" $y = x^2$

immediately clear what this picture has to do with the sets $X$ and $Y$ and the rule etc. which is supposed to be what a function is. What we really have here is not the function per se, but the **graph** of the function. Let me try to clarify. The domain set in this case is the interval $X = (-1, 1)$. In Figure 2 I've redrawn the graph and indicated the domain in its proper place. It was the idea of René Descartes to place

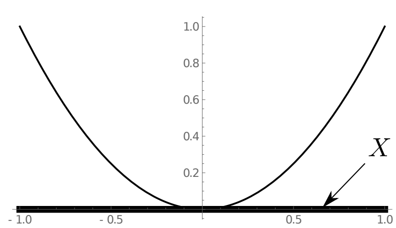


Figure 2: the domain $X = (-1, 1)$ a la (De)carte(s)

the set $Y = \mathbb{R}$ orthogonal to the domain interval $(-1, 1)$ and form the set of ordered pairs

$$\mathcal{G} = \{(x, f(x)) : x \in X\}.$$

This set, which in this case is a parabolic curve, is called the **graph** of the function. It is a subset of the two-dimensional Euclidean space $\mathbb{R}^2$, also known as the cartesian plane (after (De)cartes) or the coordinate plane.

Hopefully, the relation of the graph (of a function) to the rule (which is the function) is, more or less, clear.

## Continuity

It turns out that calculus is not just about any real valued functions of a real variable. We ususally want the functions under consideration to also satisfy some **regularity conditions**. This means, for example, we're not going to talk about the function which assigns the value 1 to every rational number between $-1$ and 1 but assigns the value 0 to every irrational number. At least we're not going to talk about that function much. But you should try to draw its graph.

We want functions which are (for starters) **continuous**. Roughly speaking continuity of a function means that you can trace your pencil across the graph of the function without lifting it from the paper. More precisely, it means that given any domain element $x_0$ and any positive number $\epsilon$, there is some (other) positive number $\delta$ such that whenever $x$ is a domain element with $|x - x_0| < \delta$, then there always holds $|f(x) - f(x_0)| < \epsilon$.[1] Another way to say this is

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon.$$

Here the symbol "$\Longrightarrow$" is read "implies." Finally, a third way to say the same thing is that for every $x_0 \in X$,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In a certain sense, continuous functions are "well-behaved" (regular) enough so that some complicated problems are ruled out. For example, any function which is continuous on an open interval $(a_0, b_0)$ can be integrated over any closed subinterval $[a, b] \subset (a_0, b_0)$. This means that if $a_0 < a < b < b_0$, then

$$\int_a^b f(x)\, dx$$

[1] See `https://www.youtube.com/watch?v=zxFCQplZgKI` and `https://www.youtube.com/watch?v=sSkIsBCbTCQ`

makes sense. We'll briefly review what this means below, but suffice it to say that continuous real valued functions of one real variable are a good place to start to try to do calculus. Most of the functions you know fall into this class. Of course, continuity is not enough to talk about differentiability. For example, the absolute value function is continuous on $(-1, 1)$, but its graph looks like the drawing in Figure 3, and there is no derivative at $x = 0$.

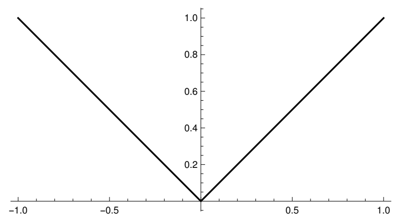


Figure 3: $f(x) = |x|$ is continuous by not differentiable at $x = 0$

Incidentally, now that we've talked about graphs, you may want to go back to the definition of a function and consider how the "to each" and "a unique" parts of the definition relate to the "vertical line test" for graphs.

# Derivatives

In spite of the fact that it requires more regularity to differentiate a function than to integrate it, we usually learn about derivatives first. Given a function $f : (a, b) \to \mathbb{R}$ and a domain element $x_0 \in (a, b)$, the **derivative** of $f$ at $x = x_0$ is given by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

whenever this limit exists. If the derivative exists at every $x_0 \in (a, b)$, we say $f$ is **differentiable** on $(a, b)$. If $f$ is differentiable, then $f$ is also continuous, though as we have pointed out, the reverse implication does not hold. The quantity

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is called a **difference quotient**. Geometrically, the difference quotient gives the slope of the secant line connecting the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ which lie on the graph of $f$ in the plane. The limit is the slope of the tangent line to the graph at $(x_0, f(x_0))$ (if it exists). David Jerison gives a nice description/explanation of why it is difficult to calculate the slope of this tangent line in any other way in his first lecture on Calculus at MIT: `https://www.youtube.com/watch?v=7K1sB05pE0A`.

Again, we can phrase the existence of the limit in terms of $\epsilon$ and $\delta$: There is a number $L$ such that for any $\epsilon > 0$, there is a $\delta > 0$ for which

$$0 < |h| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(x_0 + h) - f(x_0)}{h} - L \right| < \epsilon.$$

Notice that here we have not allowed $h = 0$ both because the difference quotient does not make sense for $h = 0$ and because that is the definition of taking a limit. And, of course, the limit $L$ is denoted by $f'(x_0)$.

The derivative also gives the **instantaneous rate of change** of the value of $f$ with respect to the variable $x$. This accounts for the relation between derivatives and velocity: If the independent variable is time, and the function gives position, then the rate of change of the position with respect to time is **velocity**. So derivatives give a kind of generalized velocity.

This is the main theoretical aspect of the derivative along with the two most important interpretations (slope of the tangent line and rate of change). Beyond that, one learns all kinds of techniques to find the value of the derivative for particular functions. There's the power rule, and the chain rule, and the quotient rule, and the derivatives for the exponential, logarithmic, and trigonometric functions. There are also manifold applications. One particular application worth mentioning is the use of the derivative to find maxima and minima of a real valued function of a real variable.

With more regularity, one can also define derivatives of higher order...like second derivatives, which tell you about the convexity of the graph.

**Exercise 1** *Let $f(x) = |x|$ be the absolute value function. Let*

$$g(x) = \int_{-1}^{x} f(x)\,dx.$$

*Show that $g$ is differentiable at all points, but $g$ is not twice differentiable. Hint: Evaluate the integral (correctly) to find another expression for $g$.*

## Integrals

As mentioned above, any continuous function can be integrated. The value of the integral is

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}).$$

This, of course, looks like a complicated mess. As with the difference quotient, the first thing to understand is the object after the limiting operation. In this case, it's called a **Riemann sum**. To make a Riemann sum, you partition your interval

$$\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b.$$

This means you take a collection of consecutive domain values $x_0, x_1, x_2, \ldots, x_n$. This breaks your interval into a sequence of subintervals $[x_{j-1}, x_j]$, and on each subinterval you take another domain value $x_j^*$. This means $x_{j-1} \le x_j^* \le x_j$. Notice that the length of each subinterval is $x_j - x_{j-1}$, so the quantity

$$f(x_j^*)(x_j - x_{j-1})$$

is the area of a rectangle with base the interval $(x_{j-1}, x_j)$ and height the function value $f(x_j^*)$, as long as $f(x_j^*) \ge 0$. If it happens that $f(x_j^*) < 0$, then you get minus the area of the rectangle with base the interval $(x_{j-1}, x_j)$ and height $|f(x_j^*)|$. Thus, the Riemann sum is a sum of (signed) areas of rectangles. Here are a couple videos from Sal Khan for you to watch if this is not clear:

> `https://www.youtube.com/watch?v=dEAk0BHBYCM`
>
> and
>
> `https://www.youtube.com/watch?v=ViqrHGae7FA`

There's also a third video that follows these you may want to watch if it's still not clear.

So if we know what a Riemann sum means, then we can talk about the limit. The limit is a limit as the "norm" of the partititon tends to zero. The norm in this case means just the maximum increment, that is

$$\|\mathcal{P}\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Thus, for the limit to exist means there is some number $L$ such that given any $\epsilon > 0$, there is some $\delta > 0$ such that if $\|\mathcal{P}\| < \delta$ (and the partition points $x_0, x_1, \ldots, x_n$ as well as the evaluation points $x_1^*, x_2^*, x_n^*$ are arbitrary subject to this condition), then we have

$$\left| \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) - L \right| < \epsilon.$$

As mentioned above, if the function is continuous, then the limit always exists. If the limit does not exist, then the function is **not Riemann integrable,** though it still may make sense to integrate it using some kind of more general approach to integration. When the limit does exist, we call it the integral of $f$ and write

$$L = \int_a^b f(x)\, dx.$$

The geometric interpretation is that if $f$ is a postive function, then this number represents the area under the graph of $f$. If $f$ is a negative valued function, then the integral is negative the area between the graph and the domain axis. If the function has both positive and negative values, then the areas above count positive and the areas below (the domain axis) count as negative, and you just add them up to see what you get; the result is a kind of signed area between the graph of the function and the domain axis.

Physically, if the domain axis is time and the function $f$ is velocity, then

$$\int_a^b f(t)\, dt$$

represents the total amount of change (or displacement) in the position at time $t = b$ starting from whatever position was taken at time $t = a$. That is, if we write $f(t) = v(t)$ for velocity, and $x = x(t)$ for position, then

$$x(b) = x(a) + \int_a^b v(t)\, dt.$$

More generally, the integral

$$\int_a^b f(x)\, dx$$

represents the **displacement** of the value of a function $F$ with respect to change in the independent variable $x$ on the interval $[a, b]$ as long as $f : [a, b] \to \mathbb{R}$ gives the rate of change of $F$ with respect to $x$.

**Exercise 2** *Verify this physical interpretation of the derivative with two examples:*

1. *A car goes sixty mph for two hours.*
2. *The temperature rises 15 degrees per hour for 6 hours starting at 40 degrees at 6 AM.*

Again, there are techniques to integrate various functions explicitly. Some easy ones are $u$-substitution and a kind of reverse version of the power rule. There's also integration by parts, which is the reverse version of the product rule.

Some other physical quantities, like center of mass, are also defined in terms of integrals, and one can compute various areas and volumes.

**Exercise 3** *Derive the formula $A = \pi r^2$ for the area of a circle of radius $r$ using integration. Bonus: Derive it without using integration.*

## The relation between differentiation and integration

This section is of somewhat less interest for our course 2550 because our course material does not really cover generalization of the relations between differentiation and integration, though we may talk about some immediate easy applications. For that reason and others, it's reasonable for me to include a short review.

The relation in 1-D calculus is called the **fundamental theorem of calculus** and it has two versions along the following lines:

**Definite integral version:** If $F : [a, b] \to \mathbb{R}$ is differentiable with $F' = f : [a, b] \to \mathbb{R}$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

This says, roughly, that if you differentiate a function first ($f = F'$) and then integrate, you get back the increment of the original function $F$. This is kind of just the "anti-derivative rule" which one learns when integrating basic functions.

The second version is somewhat more interesting.

**Indefinite integral version:** If $f : [a, b] \to \mathbb{R}$ is continuous, then the function $F : [a, b] \to \mathbb{R}$ with values defined by the rule

$$F(x) = \int_a^x f(t)\, dt$$

is a differentiable function and $F'(x) = f(x)$.

This says, roughly, that if you integrate a function first (in the sense of integrating from $a$ up to $x$), then you can differentiate **and** you get back the value of the original function $f$.

**Exercise 4** *Graph the function* $f(x) = 1/\sqrt{1-x^2}$ *on the interval* $(-1, 1)$*, i.e., draw the graph of* $f$*. Graph the function*

$$F(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt.$$

*What is the derivative of* $F$*? What is* $F$*?*

## Multivariable Calculus and Introduction to Multivariable Calculus

Multivariable Calculus is about generalizing all the topics we've discussed above by allowing the domain elements and the values of the functions under consideration to be something other than intervals in $\mathbb{R}$. The basic functions one considers are the following:

$$f : (a, b) \to \mathbb{R}^2 \qquad \text{or} \qquad \mathbb{R}^3.$$

This is called a **vector valued function of one real variable**.

$$f : U \to \mathbb{R} \qquad \text{where } U \text{ is an open subset of } \mathbb{R}^2 \text{ or } \mathbb{R}^3.$$

This is called a **real valued function of several variables**. If you don't know what an open set in a higher dimensional space is, don't worry. You can just think of $U$ as an open ball.

$$f : \Gamma \to \mathbb{R} \qquad \text{where } \Gamma \text{ is a curve in } \mathbb{R}^2 \text{ or } \mathbb{R}^3.$$

This is a **real valued function defined on a curve**. Yes, you can do calculus on curves!

$$f : \mathcal{S} \to \mathbb{R} \qquad \text{where } \mathcal{S} \text{ is a surface in } \mathbb{R}^3.$$

This is a **real valued function defined on a surface**.

The first step is to take each of these examples and explain how to differentate and integrate such a function. Actually, the first step is usually to discuss some background on dealing with the higher dimensional spaces on which these functions can be defined and take their values. One exception here is the function on a curve—that's still a one dimensional space—but still it requires some explanation, thought, and work just to think about a real valued function being defined on such a set (before one can think about differentiating and integrating it).

This background step is done partially in Chapter 12: The geometry of space. Maybe this is a good time for an outline of our course material:

1. Chapter 12: The geometry of space ($\mathbb{R}^3$ especially)

2. Chapter 13: Vector valued functions of one variable

3. Chapter 14: Differentiation of real valued functions of several variables

4. Chapter 15: Integration of real valued functions of several variables

What you should note is conspicuously missing is the discussion of differentiating and integrating on curved objects like curves and surfaces. Also missing is the discussion of generalizing the fundamental theorem of calculus. These topics are covered in MATH 2551 Multivariable Calculus.

So there you have it. That's what Multivariable Calculus is about...of course without the details. We'll cover the details of Chapters 12-15 in the course.

**Exercise 5** *Draw the graph of* $\tan^{-1}$ *the arctangent function and find its derivative. Hint:* $\tan(\tan^{-1}(x)) = x$*; apply the chain rule and draw a triangle.*

**Exercise 6** *Remember our function* $f : (-1, 1) \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational.} \end{cases}$$

*What happens if you form a Riemann sum for $f$ and take the largest value of $f$ on each subinterval of the partititon? This is called an* **upper Riemann sum**. *What happens if you take the smallest value? This is called a* **lower Riemann sum**. *What does this tell you about the limit of the Riemann sums for this function?*

**Exercise 7** *Let $U$ denote an upper Riemann sum for a function $f : (a, b) \to \mathbb{R}$ which is Riemann integrable and let $L$ denote a lower Riemann sum for the same function. Show that*

$$L \le \int_a^b f(x)\, dx \le U.$$

**Exercise 8** *Let* $B_1(0)$ *denote the open unit disk in* $\mathbb{R}^2$*. Consider the function* $\mathbf{v} : B_1(0) \to \mathbb{R}^2$ *by* $\mathbf{v}(x_1, x_2) = (x_1, x_2)$*. Can you draw a picture which illustrates what this function is doing, i.e., what the rule means? Note: This is not one of the kinds of functions we mentioned above. It's called a* **vector valued function of a vector variable** *or a* **vector field***. These are also important kinds of functions in MATH 2550 and 2551, but we don't really integrate or differentiate them too much—though it can be done. We'll get to that later.*

# Administrative comments

This course, *Introduction to Multivariable Calculus,* is a "two hour" course, meaning the students get only two hours of administrative course credit for taking it. As I observe the course catalogue, I see that the old **Calculus III: Multivariable Calculus**, which was a "four hour" course, has been replaced by two courses: *Introduction to Multivariable Calculus* (the "two hour" course I am teaching) and *Multivariable Calculus*, another "four hour" course. Apparently, someone thought the old Multivariable Calculus included too much material for a one semester course, or at least the students (on average) were not ready to learn the material that quickly. That's an interesting perspective, and I'm not sure what I think about it. On the one hand, splitting the material into two parts with...