Matters of Moment

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It may be helpful to have a discussion of the moments of an extended body which is more general and a little more physically motivated than the one in the Thomas Calculus text.

Preliminaries: Kinetic Energy of a Moving Point Mass

If a certain mass m is concentrated at a point $\mathbf{x}$ (an idealization called a **point mass**), then the associated **kinetic energy** is said to be given by

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}|^2 \qquad \text{where} \qquad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{1}$$

This kinetic energy is sometimes called the **translational kinetic energy**.

As a special case, we can imagine a point mass rotating with angular velocity ω about an axis. Let's say the axis is along a line $\ell(t) = \mathbf{p}_0 + t\mathbf{w}$ through a point $\mathbf{p}_0$ in the direction $\mathbf{w}$, and the rotating point mass is rotating in a plane orthogonal to this axis spanned by unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which are orthogonal to $\mathbf{w}$. In this case, we should have

$$\mathbf{x}(t) = \mathbf{p}_0 + t_0\mathbf{w} + \mathbf{r}(t) = \mathbf{p}_0 + t_0\mathbf{w} + r(\cos(\theta_0 + \omega t), \sin(\theta_0 + \omega t)) \tag{2}$$

where $\mathbf{r} = \mathbf{r}(t) = r(\cos(\theta_0 + \omega t), \sin(\theta_0 + \omega t))$ is a vector valued function of constant modulus $r = |\mathbf{r}|$, and θ_0 is some initial angle with respect to the plane $\{\mathbf{p}_0 + t_0\mathbf{w} + \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 : (\alpha, \beta) \in \mathbb{R}^2\}$. From the expression (2), the velocity and kinetic energy are easy to compute:

$$\frac{d\mathbf{x}}{dt} = \omega r(-\sin(\theta_0 + \omega t), \cos(\theta_0 + \omega t)) \quad \text{and} \quad \text{K.E.} = \frac{1}{2}mr^2\omega^2.$$

It will be noted that this expression for the **rotational kinetic energy** can be interpreted as having the same form as the translational kinetic energy but with the **linear velocity** $|\mathbf{v}|$ replaced by the angular velocity ω and the mass replaced by the quantity mr^2 which depends on the radial distance r from the axis. The notion of **moments** is very much motivated, if not based, on this kind of **parallel structure** interpretation of quantities.

Before we move on from point masses, let us make a couple simple observations. If we have a finite number of point masses, say two of them m_1 and m_2, with positions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively, then it makes good sense to consider the **total kinetic energy** of the **system** composed of these masses. The translational kinetic energy is just the sum

$$\frac{1}{2}m_1|\mathbf{v}_1|^2 + \frac{1}{2}m_2|\mathbf{v}_2|^2.$$

This holds no matter how the point masses may be moving in space. As we have described it, the rotational kinetic energy of a particle only makes sense with respect to a given axis. The axis may move (if the object is translating for example), but the expression we have used above requires the relation $(\mathbf{x} - \mathbf{q}_0) \cdot \mathbf{w} = 0$ for some point $\mathbf{q}_0 = \mathbf{p}_0 + t_0\mathbf{w}$ in the axis of rotation. Consequently, we must assume our system of point masses is coherently rotating about a single axis. (Technically, we could assume a different axis for each point mass, but we definitely need rotation of each mass about some axis.) Under this **coherence of rotation** it makes sense to talk about the total rotational kinetic energy, and the value is again just a sum

$$\frac{1}{2}m_1 r_1^2\omega_1^2 + \frac{1}{2}m_2 r_2^2\omega_2^2.$$

Distributed Mass: Density

A more realistic representation of a physical object which can move through space is given by assigning a density function $\rho : \mathcal{V} \to [0, \infty)$ to a **geometric volume** $\mathcal{V}$ in space. The units of a quantity can be useful to consider and are often denoted by putting square brackets around the quantity:

$$[\rho] = \frac{M}{L^3}.$$

This means that **density** is measured in units of mass per volume. (Volume has units length cubed.) In such a situation, the total mass is obtained as the limit of a

Riemann sum, that is to say, an integral:

$$m = \lim_{\|\mathcal{P}\| \to 0} \sum_j \rho(\mathbf{x}_j^*) \operatorname{vol}(\mathcal{V}_j) = \int_{\mathcal{V}} \rho$$

where $\|\mathcal{P}\|$ is the norm of a partititon $\mathcal{P} = \{\mathcal{V}_j\}$ of the volume $\mathcal{V}$ as usual. Similarly, the total kinetic energy of a translating object modeled in this way is the limit of the sum of the kinetic energies associated with the small portions $\mathcal{V}_j$ of a partition $\mathcal{P}$ of $\mathcal{V}$:

$$\text{K.E.} = \lim_{\|\mathcal{P}\| \to 0} \sum_j \frac{1}{2} \rho(\mathbf{x}_j^*) \operatorname{vol}(\mathcal{V}_j) |\mathbf{v}|^2 = \frac{1}{2} \int_{\mathcal{V}} \rho |\mathbf{v}|^2$$

so we obtain the same formula (1). This is under the assumption that every portion of the object is moving with the same linear velocity $\mathbf{v}$ and that kinetic energy associated with the small volume $\mathcal{V}_j$ is approximately that of a point mass $m_j = \rho(\mathbf{x}_j^*) \operatorname{vol}(\mathcal{V}_j)$ translating with that velocity.

The model we have described using a volume with a density is called an **extended body**, and the density is called a **volumetric density**. The same idea can be applied to what is called a **lamina** which is a two dimensional version of the same thing. In the case of a lamina, one is given a planar object (or a surface) with a density ρ with units

$$[\rho] = \frac{M}{L^2}.$$

A density with these units is called an **areal density**. The mass of a lamina is, of course, given by

$$m = \int_{\mathcal{U}} \rho.$$

(Incidentally, the mass computed in either of these ways is called a zero order moment of the extended body or lamina.)

We can try to make the same calculation of rotational kinetic energy for an extended body. Let us assume that every portion of the extended body is rotating with angular velocity ω about the same axis. In particular, each piece $\mathcal{V}_j$ of a partition has kinetic energy approximately that of a point mass $m_j = \rho(\mathbf{x}_j^*) \operatorname{vol}(\mathcal{V}_j)$ with position $\mathbf{x}_j^*$ rotating about the axis with angular velocity ω, that is,

$$\frac{1}{2} m_j r_j^2 \omega^2 = \frac{1}{2} \rho(\mathbf{x}_j) \operatorname{vol}(\mathcal{V}_j) r_j^2 \omega^2$$

where $r_j^* = |\mathbf{x}_j^* - \mathbf{q}_j^*|$ is the radial distance from $\mathbf{x}_j^*$ to the axis of rotation. Taking a sum and the limit as the norm of the partition tends to zero, we find

$$\text{K.E.} = \frac{1}{2}\int_{\mathcal{V}} r^2\rho\,\omega^2.$$

Thus, for extended bodies, the mass is replaced by something more complicated.

$$\text{K.E.} = \frac{1}{2}I\,\omega^2 \qquad \text{where} \qquad I = \int_{\mathcal{V}} r^2\rho.$$

The quantity I is called the **moment of inertia** or second moment of the extended body with respect to its axis of rotation. This quantity takes the place of the mass (when the angular velocoity ω takes the place of the linear velocity $|\mathbf{v}|$). It should be noted that the radius of rotation r is a function defined on the volume, or extended body, $\mathcal{V}$ representing the object. So the integrand is a product of two functions $r(\mathbf{x})^2$ and $\rho(\mathbf{x})$.

Force and Newton's Law

You might notice that we talked about a **zero moment** (i.e., the mass of an extended body or lamina) and the **second moment** of an extended body, and the discussion of the second moment can be easily extended to apply to a lamina as well. You might be wondering about the **first moment**.

Remember that Newton's second law asserts $\mathbf{F} = m\mathbf{a}$. We usually think of this as applied to a point mass with position $\mathbf{x}$ so that it reads

$$\mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2}. \tag{3}$$

Our objective is to find an appropriate version of Newton's second law, applying to an extended body, which demonstrates parallel structure with (3).

It makes sense to add up forces over an extended body in the following context: Say you have a **force density field** which is a vector field $\mathbf{G}$ with units

$$[\mathbf{G}] = \frac{F}{M} = \frac{L}{T^2}$$

where F represents "force" and has units ML/T^2. You have this kind of thing with the gravity field $\mathbf{G} = (0, 0, -g)$. So if you take a point mass m in space (near the earth) it experiences a downward gravitational force $\mathbf{F} = (0, 0, -mg)$. More generally, if you have such a force field in space, then it makes sense to add up the forces on the partition pieces of an extended object to get an approximate total force on the object:

$$\sum_j \rho(\mathbf{x}_j^*)\,\mathrm{vol}(\mathcal{V}_j)\mathbf{G}(\mathbf{x}_*). \tag{4}$$

Taking the limit, we get an integral quantity representing the total force on the object:

$$\mathbf{F} = \int_{\mathcal{V}} \rho\mathbf{G}.$$

On the other hand, we can apply Newton's second law to each piece to get an approximate equation

$$\rho(\mathbf{x}_j^*)\,\mathrm{vol}(\mathcal{V}_j)\mathbf{G}(\mathbf{x}_*) \approx \rho(\mathbf{x}_j^*)\,\mathrm{vol}(\mathcal{V}_j)\frac{d^2\mathbf{x}_j^*}{dt^2}.$$

Exercise 1 *Substitute this expression into the Riemann sum (4) and conclude that*

$$\mathbf{F} = \frac{d^2}{dt^2}\int_{\mathcal{V}} \mathbf{x}\rho. \tag{5}$$

The components of the integral

$$\int_{\mathcal{V}} \mathbf{x}\rho$$

are called the **first moments** *of the extended body. Write down the three first moments of an extended body, and use (5) along with familiar formulas to give an extended body interpretation with a structure parallel to (3). Hint: Define $\bar{\mathbf{x}}$ appropriately so that (5) takes the form*

$$\mathbf{F} = m\frac{d^2\bar{\mathbf{x}}}{dt^2}.$$