Motion on a (Potential) Surface in Gravity

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We wish to consider a point mass with motion constrained to a given surface under the influence of gravity and having a given initial position and velocity as indicated in Figure 1. This problem uses many elementary techniques from multivariable calculus.

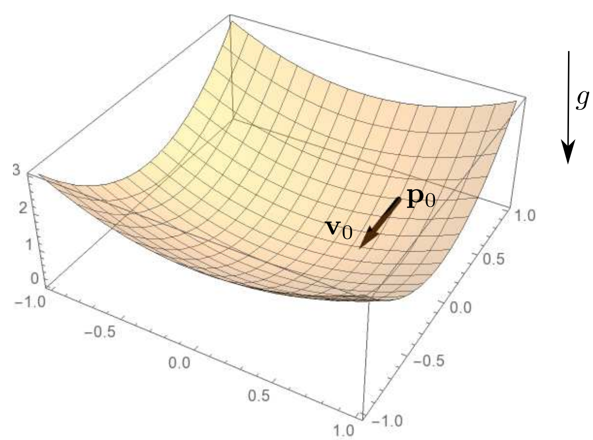


Figure 1: Motion on a surface

In addition, some aspects of surface theory (the differential geometry of curves and surfaces) beyond those usually covered in multivariable calculus are required. Finally, the problem provides a natural introduction to some basic considerations of ordinary differential equations and the numerical solution of those equations.

Setting up the problem

Let $\mathcal{S}$ be a surface given as the graph of a (smooth) function $u : \mathbb{R}^2 \to \mathbb{R}$. Let $\mathbf{v}_0$ be a vector tangent to $\mathcal{S}$ at $(\mathbf{x}_0, u(\mathbf{x}_0)) = (x_0, y_0, u(x_0, y_0))$. Assume a point mass m moves on $\mathcal{S}$ subject to gravitational acceleration $-g\mathbf{e}_3 = (0, 0, -g)$ and "friction." Under these conditions, we would like to find the path $\mathbf{r} : [0, T) \to \mathcal{S}$ of the point mass (on whatever time interval $[0, T)$ we are able to find it). As stated, modeling the friction force is part of the problem.

The basic idea is to use Newton's second law:

$$m\mathbf{r}'' = \mathbb{F} \tag{1}$$

where $\mathbb{F}$ is the sum of the forces acting on the point mass. It is, of course, necessary to consider what it means for the motion to be constrained to the surface $\mathcal{S}$.

Motion constrained to a surface

We can write $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), u(x(t), y(t)))$, and it is enough to determine the two real valued functions $x = x(t)$ and $y = y(t)$. Note that using this form

$$\mathbf{v} = \mathbf{r}' = \frac{d\mathbf{r}}{dt} = (x', y', u_x x' + u_y y')$$

where

$$u_x = \frac{\partial u}{\partial x}(x(t), y(t)) \quad \text{and} \quad u_y = \frac{\partial u}{\partial y}(x(t), y(t)).$$

Similarly,

$$\mathbf{a} = \mathbf{r}'' = (x'', y'', u_x x'' + u_{xx} x'^2 + u_{xy} x' y' + u_y y'' + u_{xy} x' y' + u_{yy} y'^2).$$

This may be used in the expression on the left in Newton's law (1).

Forces

We are given one force $\mathbb{F}_g = -g\mathbf{e}_3 = (0, 0, -g)$. Without this force there would still be an interesting motion, so we will also want to consider the case $g = 0$.

Perhaps the most interesting force is what a physicist might call the **reaction force** exerted by the surface when the mass produces a force on the surface in consequence of its motion. This force is naturally considered in an orthonormal frame adapted to the surface $\mathcal{S}$ (and **adapted frame**) which may be obtained as follows:

At each point $\mathbf{p} = (\mathbf{x}, u(\mathbf{x})) \in \mathcal{S}$ we may consider a curve

$$\gamma_1(t) = (\mathbf{x} + t\mathbf{e}_1, u(\mathbf{x} + t\mathbf{e}_1))$$

which projects to a line parallel to the x-axis. We may then obtain a vector $\gamma_1'(0)$ tangent to $\mathcal{S}$ at $(\mathbf{x}, u(\mathbf{x}))$. This vector is

$$\gamma_1'(0) = (1, 0, u_x(\mathbf{x})). \tag{2}$$

Exercise 1 *Calculate the third component*

$$\frac{d}{dt}u(\mathbf{x} + t\mathbf{e}_1)\Big|_{t=0}$$

in (2).

Similarly, we can get a second tangent vector by considering

$$\gamma_2(t) = (\mathbf{x} + t\mathbf{e}_2, u(\mathbf{x} + t\mathbf{e}_2)) :$$

$$\gamma_2'(0) = (0, 1, u_y(\mathbf{x})).$$

The tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ constitute a **basis** for the tangent plane to $\mathcal{S}$ at $\mathbf{p} = (\mathbf{x}, u(\mathbf{x}))$. This means every displacement vector $\mathbf{w}$ in that plane may be expressed uniquely as a linear combination $\mathbf{w} = a\gamma_1'(0) + b\gamma_2'(0)$. This basis is not (usually) an orthonormal basis, so figuring out the coefficents a and b for a given tanent vector $\mathbf{w}$ may be difficult.

Exercise 2 *When are $\gamma_1'(0)$ and $\gamma_2'(0)$ orthonormal?*

There are two obvious things we can do with the basis $\{\gamma_1'(0), \gamma_2'(0)\}$:

1. We can construct an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for the tangent plane $T_{\mathbf{p}}\mathcal{S}$.

2. We can find a normal $\mathbf{N}$ to $\mathcal{S}$ so that $\{\gamma_1'(0), \gamma_2'(0), \mathbf{N}\}$ is a basis for all of $\mathbb{R}^3$.

If we accomplish both of these things and if $\mathbf{N}$ has unit length (which is easy to arrange), then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$ will be our adapted frame at $\mathbf{p}$ as shown in Figure 2.

Let's start by finding a normal to the surface.

$$\gamma_1'(0) \times \gamma_2'(0) = \begin{pmatrix} 1 \\ 0 \\ u_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ u_y \end{pmatrix} = \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}.$$

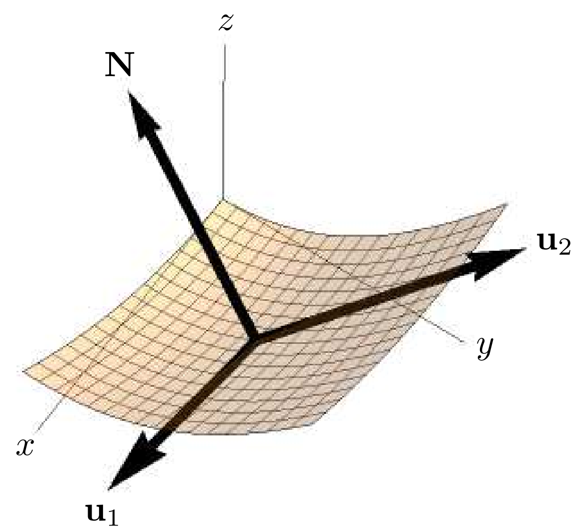


Figure 2: Adapted orthonormal frame on a surface

We say this is an **upward normal** because the third component is positive. We get the **upward unit normal**

$$\mathbf{N} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + u_x^2 + u_y^2}}. \tag{3}$$

Remember that $u = u(x, y)$ is given, so $\mathbf{N}$ can be considered as a given vector at each point of $\mathcal{S}$.

Next we can exchange $\gamma_1'(0)$ for a unit vector

$$\mathbf{u}_1 = \frac{\gamma_1'(0)}{|\gamma_1'(0)|} = \frac{(1, 0, u_x)}{\sqrt{1 + u_x^2}}. \tag{4}$$

Finally, we want a (unit) vector $\mathbf{u}_2$ tangent to $\mathcal{S}$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$ is a right-handed orthonormal basis for $\mathbb{R}^3$ at $\mathbf{p} = (\mathbf{x}, u(\mathbf{x}))$. The right-handed condition means $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{N}$. I can think of two ways to find $\mathbf{u}_2$. Perhaps the easiest is to set

$$\mathbf{u}_2 = \mathbf{N} \times \mathbf{u}_1.$$

This computation leads to

$$\mathbf{u}_2 = \frac{(-u_x u_y, 1 + u_x^2, u_y)}{\sqrt{(1 + u_x^2)(1 + u_x^2 + u_y^2)}}. \tag{5}$$

This is not the prettiest formula, but it should work.

Exercise 3 *Check directly that* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$ *is a right-handed orthonormal basis.*

Exercise 4 *Find* $\mathbf{u}_2$ *in a different way:*

1. *Take* $\gamma_2'(0) = (0, 1, u_y)$ *and project it onto* $\mathbf{u}_1 = \gamma_1'(0)/|\gamma_1'(0)|$*:*

$$(\gamma_2'(0) \cdot \mathbf{u}_1)\, \mathbf{u}_1.$$

2. *Check that "the rest" of* $\gamma_2'(0)$*, i.e.,*

$$\gamma 2'(0) - (\gamma_2'(0) \cdot \mathbf{u}_1)\, \mathbf{u}_1,$$

is a tangent vector orthogonal to $\mathbf{u}_1$*.*

3. *Set*

$$\mathbf{u}_2 = \frac{\gamma 2'(0) - (\gamma_2'(0) \cdot \mathbf{u}_1)\, \mathbf{u}_1}{|\gamma 2'(0) - (\gamma_2'(0) \cdot \mathbf{u}_1)\, \mathbf{u}_1|}. \tag{6}$$

4. *Check that* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$ *is a right-handed orthonormal frame and is the same one we got by the other method.*

This second procedure may be familiar as **Gramm-Schmidt orthonormalization** *from linear algebra.*

We have now obtained our **adapted orthonormal frame** $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$. See formulas (4), (5), and (3). For (5) we may also use (6). The big advantage of this frame is that any vector $\mathbf{w} \in \mathbb{R}^3$ can be written easily (and uniquely) as

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{w} \cdot \mathbf{u}_2)\, \mathbf{u}_2 + (\mathbf{w} \cdot \mathbf{N})\, \mathbf{N}.$$

This will turn out to be really important.

Reaction force

One immediate application of our analysis of the geometry of $\mathcal{S}$ is that we can express the reaction force as

$$\mathbb{F}_N = \phi \mathbf{N}$$

where the function ϕ will adjust to cancel any other force (or component of a force) orthogonal to $\mathcal{S}$.

Friction

The velocity $\mathbf{v} = \mathbf{r}' = (\mathbf{x}', Du \cdot \mathbf{x}') = (x', y', u_x x' + u_y y')$ computed above is always tangent to $\mathcal{S}$. Perhaps the simplest way to model friction is to write

$$\mathbb{F}_f = -\mu \mathbf{v}$$

where μ is a positive constant (or a non-negative constant if we want to include the case where there is no friction). This means friction is always proportional to, and in the opposite direction of the velocity. There may be other ways to model friction, but this is one simple way.

The Motion

We are now ready to reassemble our forces in Newton's law and see what we get.

$$m\mathbf{r}'' = \mathbb{F}_g + \mathbb{F}_N + \mathbb{F}_f = -g\mathbf{e}_3 + \phi\mathbf{N} - \mu\mathbf{r}'.$$

We will decompose both the left side of this equation and each of the component forces in terms of the orthonormal frame. First of all

$$\mathbf{r}'' = (\mathbf{r}'' \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{r}'' \cdot \mathbf{u}_2)\, \mathbf{u}_2 + (\mathbf{r}'' \cdot \mathbf{N})\, \mathbf{N},$$

and we get similar expressions for the forces on the right. Equating the components with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{N}\}$, we get three scalar equations:

$$\begin{aligned}
m(\mathbf{r}'' \cdot \mathbf{u}_1) + \mu(\mathbf{r}' \cdot \mathbf{u}_1) + g(\mathbf{e}_3 \cdot \mathbf{u}_1) &= (m\mathbf{r}'' + \mu\mathbf{r}' + g\mathbf{e}_3) \cdot \mathbf{u}_1 = 0, \\
m(\mathbf{r}'' \cdot \mathbf{u}_2) + \mu(\mathbf{r}' \cdot \mathbf{u}_2) + g(\mathbf{e}_3 \cdot \mathbf{u}_2) &= (m\mathbf{r}'' + \mu\mathbf{r}' + g\mathbf{e}_3) \cdot \mathbf{u}_2 = 0, \\
m(\mathbf{r}'' \cdot \mathbf{N}) + g(\mathbf{e}_3 \cdot \mathbf{N}) &= (m\mathbf{r}'' + g\mathbf{e}_3) \cdot \mathbf{N} = \phi(\mathbf{N} \cdot \mathbf{N}) = \phi.
\end{aligned}$$

The last equation is not so important for us unless we want to know about the stresses on the surface; this equation may be taken as a definition of ϕ (the normal force) after we know $\mathbf{r}$.

We should be able to use the first two equations to find $\mathbf{r}$. Let's write them out in more detail to see what we've got. After clearing the denominator $\sqrt{1 + u_x^2 + u_y^2}$, the first term in the first equation is

$$\begin{aligned}
m\mathbf{r}'' \cdot (1, 0, u_x) &= m[x'' + u_x(u_x x'' + u_y y'' + u_{xx} x'^2 + 2u_{xy} x' y' + u_{yy} y'^2)] \\
&= m[(1 + u_x^2)x'' + u_x u_y y'' + u_x(u_{xx} x'^2 + 2u_{xy} x' y' + u_{yy} y'^2)].
\end{aligned}$$

The second term is linear in the first derivatives

$$\mu[x' + u_x(u_x x' + u_y y')] = \mu[(1 + u_x^2)x' + u_x u_y y'].$$

And finally, the gravitational contribution to this equation becomes simply gu_x. Putting the equation back together, we find

$$m[(1+u_x^2)x''+u_x u_y y'']+\mu[(1+u_x^2)x'+u_x u_y y']+mu_x(u_{xx}x'^2+2u_{xy}x'y'+u_{yy}y'^2)+gu_x = 0.$$

We recall that $\mathbf{u}_2$ from (5) has a more complicated form involving $(-u_x u_y, 1+u_x^2, u_y)$, so we should calculate each term as before:

$$\begin{aligned} m\mathbf{r}'' \cdot (-u_x u_y, 1 + u_x^2, u_y) &= m[-u_x u_y x'' + (1 + u_x^2)y'' \\ &\qquad + u_y(u_x x'' + u_y y'' + u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2)] \\ &= m[(1 + u_x^2 + u_y^2)y'' + u_y(u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2)]. \end{aligned}$$

For the friction term we get

$$\mu[-u_x u_y x' + (1 + u_x^2)y' + u_y(u_x x' + u_y y')] = \mu(1 + u_x^2 + u_y^2)y'$$

which is again linear in the first order unknowns x' and y'. Thus, we have a second equation:

$$m(1 + u_x^2 + u_y^2)y'' + \mu(1 + u_x^2 + u_y^2)y' + mu_y(u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2) + gu_y = 0.$$

We have now obtained a system of two second order ordinary differential equations for $x = x(t)$ and $y = y(t)$:

$$\begin{cases} m[(1 + u_x^2)x'' + u_x u_y y''] + \mu[(1 + u_x^2)x' + u_x u_y y'] \\ \qquad\qquad +mu_x(u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2) + gu_x = 0 \\ m(1 + u_x^2 + u_y^2)y'' + \mu(1 + u_x^2 + u_y^2)y' \\ \qquad\qquad +mu_y(u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2) + gu_y = 0. \end{cases}$$

We have also appropriate initial conditions $x(0) = x_0$, $y(0) = y_0$, $x'(0) = x_0'$, and $y'(0) = y_0'$ where $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{v}_0 = (x_0', y_0', u_x(x_0, y_0)x_0' + u_y(x_0, y_0)y_0')$. Perhaps the most obvious and simple thing to do is solve this system numerically, or rather implement a numerical solver from a mathematical software package like Mathematica or Maple to solve the system of equations for particular initial values and plot some solutions.

It is probably not easy to solve this system explicitly, but there may be some techniques that can be applied in special cases.

Exercise 5 *The* **kinetic energy** *of the point mass (at any given time) is*

$$\text{K.E.} = \frac{1}{2}m(x'^2 + y'^2).$$

1. *Find an expression for the* **potential energy** *of the point mass (at any given time) based on the principle that the potential energy is given by a work integral*

   $$\text{P.E.} = -\int_0^{u(x,y)} \mathbb{F}_g \cdot \mathbf{e}_3 \, d\xi = gu.$$

   Here we note that

   $$\text{work} = (\text{force}) \times (\text{displacementdistance}).$$

2. *Show that if there is no friction ($\mu = 0$), then*

   $$\frac{d}{dt}[(\text{K.E.} + \text{P.E.}] = 0, \tag{7}$$

   i.e., the **total energy** *of the system is conserved during any particular motion.*

3. *In the special case where there is no friction, we can integrate (7) to obtain a first order differential relation*

   $$\frac{m}{2}(x'^2 + y'^2) + gu(x, y) = c$$

   where c is a constant.

The calculation (7) has several interesting consequences which are difficult to see (and explain) in the context of our original problem. For this reason and others, I want to consider a fundamentally simpler version of our problem.

Motion on a curve

Let $\mathcal{C}$ be a curve given as the graph of a (smooth) function $u : \mathbb{R}^1 \to \mathbb{R}$. Let $\mathbf{v}_0 = (x_0', u'(x_0)x_0')$ be a vector tangent to $\mathcal{C}$ at $(x_0, u(x_0))$. Assume a point mass m moves on $\mathcal{C}$ subject to gravitational acceleration $\mathbb{F}_g = -mg\mathbf{e}_2 = (0, -mg)$ and a friction force $\mathbb{F}_f$. Can we find the path $\mathbf{r} = \mathbf{r}(t)$ of the mass?

Again, we'll use Newton's second law $m\mathbf{r}'' = \mathbb{F}$ but this time in two dimensions with $\mathbf{r} = (x(t), u(x(t)))$, so essentially, there should be only one function $x = x(t)$ to find. For the left derivatives of $\mathbf{r}$ we have

$$\mathbf{v} = \mathbf{r}' = (x', u'x') \qquad \text{and} \qquad \mathbf{a} = \mathbf{r}'' = (x'', u'x'' + u''x'^2)$$

where it is understood that derivatives of u are with respect to x and evaluated at $x = x(t)$ and derivatives of x are with respect to time t.

The adapted frame we use in this case is composed of the unit tangent vector and the upward normal to the curve:

$$\mathbf{u} = \mathbb{T} = \frac{(1, u')}{\sqrt{1 + u'^2}} \qquad \text{and} \qquad \mathbf{N} = \frac{(-u', 1)}{\sqrt{1 + u'^2}}.$$

This frame should be familiar. Writing the acceleration vector in terms of the frame we have

$$\mathbf{r}'' = (\mathbf{r}'' \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{r}'' \cdot \mathbf{N})\,\mathbf{N} = \frac{(1 + u'^2)x'' + u'u''x'^2}{1 + u'^2}\,(1, u') + \frac{u''x'^2}{1 + u'^2}(-u', 1).$$

The gravitational force is $\mathbb{F}_g = -g\mathbf{e}_2 = (0, -g)$. Therefore, the decomposition is

$$\mathbb{F}_g = (\mathbb{F}_g \cdot \mathbf{u})\,\mathbf{u} + \mathbb{F}_g \cdot \mathbf{N})\,\mathbf{N} = -\frac{gu'}{1 + u'^2}\,(1, u') - \frac{g}{1 + u'^2}(-u', 1).$$

As before, we take $\mathbb{F}_N = -\phi\mathbf{N}$ for the reaction force from the curve, and $\mathbb{F}_f = -\mu\mathbf{v}$ for the friction force. That is,

$$\mathbb{F}_f = -\mu\frac{(1 + u'^2)x'}{1 + u'^2}\,(1, u') = -\mu x'\,(1, u').$$

Consequently, the tangent component equation is

$$m[(1 + u'^2)x'' + u'u''x'^2] + \mu(1 + u'^2)x' + gu' = 0, \tag{8}$$

which is a second order ordinary differential equation for $x = x(t)$. The normal component equation we can ignore.

Exercise 6 *Couple the equation (8) with the initial conditions $x(0) = x_0$ and $x'(0) = x'_0$ and solve this equation numerically using a mathematical software package. If you are using mathematica, you can use* `NDSolve`*. If you are using Matlab, the* *corresponding function is* `ODE45`*. Both are adaptive fourth and fifth order Runge-Kutta solvers. Alternatives software packages are Maple and Octave. You may need to express the second order equation above as an equivalent system of first order equations:*

$$\begin{cases} x' = z, & x(0) = x_0 \\ \\ z' = -\dfrac{1}{m(1+u'(x)^2)}[mu'(x)u''(x)z^2 + gu'(x)] - \dfrac{\mu}{m}z, & z(0) = x'_0. \end{cases}$$

You should be able to produce a nice animation illustrating the motion for a given curve $\mathcal{C}$ given as the graph of $u = u(x)$ and given initial conditions.

Let's also follow through with the energy considerations in this case. The kinetic energy is

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}|^2 = \frac{m}{2}(1+u'^2)x'^2,$$

and the potential energy is (up to an additive constant)

$$\text{P.E.} = -\int_0^u \mathbb{F}_g \cdot \mathbf{e}_2 d\zeta = gu.$$

Thus, we claim that with $\mu = 0$, the total energy which we can write as

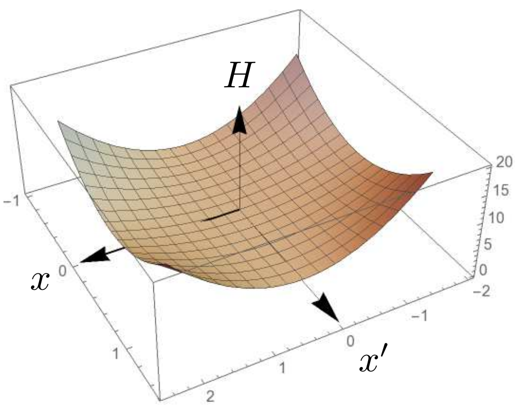
$$H(x,z) = \frac{m}{2}(1+u'(x)^2)z^2 + gu(x)$$

should be conserved. In fact, setting $x = x(t)$ and $z = x'(t)$, we find

$$\begin{aligned} \frac{d}{dt}H(x,z) &= mu'(x)u''(x)x'x'^2 + m(1+u'(x)^2)x'x'' + gu'(x)x' \\ &= mu'u''z^3 + m(1+u'^2)zz' + gu'z \\ &= mu'u''z^3 + m(1+u'^2)z\left(-\frac{1}{m(1+u'^2)}[mu'u''z^2 + gu']\right) + gu'z \\ &= mu'u''z^3 - mu'u''z^3 - gu'z + gu'z \\ &= 0. \end{aligned}$$

This means that corresponding to each solution $x = x(t)$, there is a parameterized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (x(t), z(t)) = (x(t), x'(t))$, and this curve has the property that it a level curve of the function $H : \mathbb{R}^2 \to \mathbb{R}$. The plane $\mathbb{R}^2$, when considered this way as the x, x'-plane is called **phase space** and the function H is called a **Hamiltonian conserved quantity** or a **Hamiltonian** for short. It is not too difficult to visualize the graph $\mathcal{G}$ of H and the level curves at least in simple cases. In each case, the intersection of the graph with the plane $z = 0$ is just the graph of gu over the x axis. From each point on this curve $\mathcal{G}_0 = \{(x, 0, gu(x)) : x \in \mathbb{R}\}$ there is an upwardly opening parabola in the plane parallel to the z, H-plane through $(x, 0, gu(x))$. In fact, $(x, 0, gu(x))$ is the vertex of the parabola.

The graph of H is shown in the special cases $u(x) = x^2$ and $u(x) = \cos x$ in Figure 3. The path of a curve $\gamma(t) = (x(t), x'(t))$ in phase space is called an **orbit**,



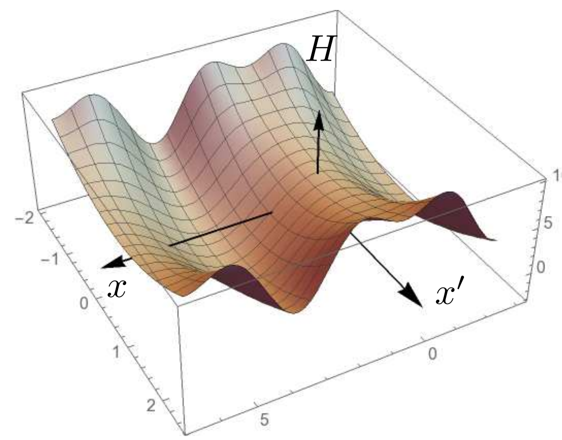


Figure 3: Hamiltonians for motion on the graph of $u(x) = x^2$ (left) and $u(x) = \cos x$ (right)

and these curves obviously contain all information about the solution x. We can conclude, in particular, that all solutions when $u(x) = x^2$ are **periodic**. This is the case, because the Hamiltonian is "bowl shaped" with a unique minimum (and unique critical point) at the origin. Thus, the level curves are simple closed curves and the orbits are periodic. This piece of information is somewhat difficult to obtain by other methods.

We noted above that the conservation of energy gives one a first order ordinary differential relation via integration. In this case, we get

$$\frac{m}{2}(1 + u'(x)^2)(x')^2 + gu(x) = c$$

where the constant $c = H(x_0, x_0')$ depends on the initial conditions. This is a **separable ordinary differential equation**, and we can "solve" it as follows: We first rearrange the equation as

$$(x')^2 = \frac{2}{m}\;\frac{c - gu(x)}{1 + u'(x)^2},$$

or

$$\frac{1 + u'(x)^2}{c - gu(x)}\;(x')^2 = \frac{2}{m}.$$

It follows that $c - gu(x)$ is non-negative on any given solution, and therefore,

$$\sqrt{\frac{1 + u'(x)^2}{c - gu(x)}}\; x' = \pm\sqrt{2}m.$$

Integrating both sides from time $t = 0$ to time t and changing variables $\xi = x(t)$ in the integral on the left, we have

$$\int_0^{x(t)} \sqrt{\frac{1 + u'(\xi)^2}{c - gu(\xi)}}\, d\xi = \pm\sqrt{2}m\; t.$$

where the plus sign is taken on portions of the solution for which $x'(t) \geq 0$ and the minus sign is taken on portions of the solution where $x'(t) \leq 0$. Notice the "solution" is given **implicitly**, since it appears in the limit of integration in the integral on the left (and only there). This means, if we want to understand the solution, we should study the invertible function $\Psi : [0, \infty) \to \mathbb{R}$ given by

$$\Psi(X) = \int_0^{X} \sqrt{\frac{1 + u'(\xi)^2}{c - gu(\xi)}}\, d\xi.$$

In terms of this function

$$x(t) = \Psi^{-1}\left(\pm\sqrt{2}m\; t\right).$$

So, there is an explicit solution of sorts, and I believe one can show directly the solutions are periodic from this expression.

Exercise 7 *We noted, under the assumption that there is a well-defined motion on a curve determined by Newton's second law, that we must have $c - gu(x) \geq 0$. Can you show directly that this condition always holds for any initial values? In particular, given an initial condition for which $c = H(x_0, x_0')$ satisfies $c - gu(x_0) > 0$, how do you know that there is not some positive time t for which c is less than the maximum value of $gu(x) = gu(x(t))$?*

Exercise 8 *Show that if friction is included (at least the kind of friction we have modeled), then*

$$\frac{d}{dt}H(x(t), x'(t)) \leq 0.$$

What does this tell you about the orbits when friction is included? In this case, we do not have a Hamiltonian conserved quantity, but the function $H = H(x, z)$ is called a **Liapunov function***.*

Exercise 9 *Implement both the equations for motion on a curve under the influence of gravity and motion on a surface numerically and produce figures like those shown in Figures 4 and 5 as well as an animation in the case of motion on a curve. One should also be able to do an animation for motion on a surface.*

Figure 3.

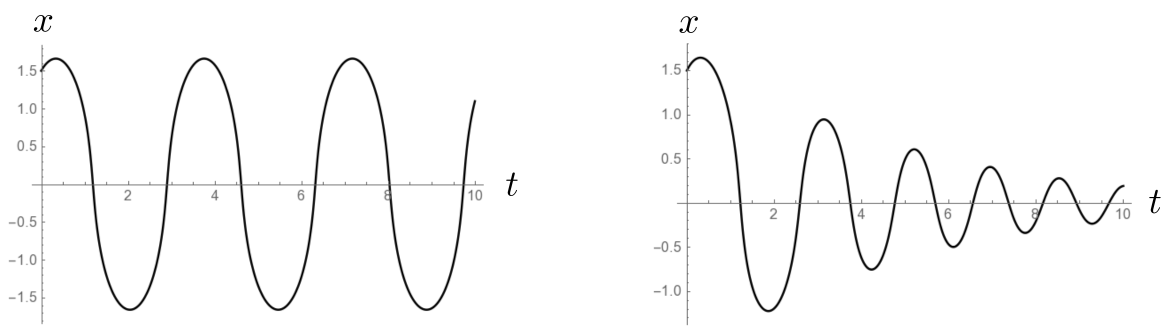


Figure 4: $x = x(t)$ for motion on the graph of $u(x) = x^2$ without friction (left) and with friction (right)

Exercise 10 *Rework these problems for motion on a curve and a surface under the influence of gravity but with different forms for the friction. In particular, it is natural to consider a friction contribution which is proportional to the normal force. This will require careful consideration of the derived expression for $\mathbb{F}_N = \phi N$.*

Research Problem

It was my initial objective to produce a figure somewhat like that illustrated in Figure 5. In particular, I wanted to produce a motion with oscillations in one direction

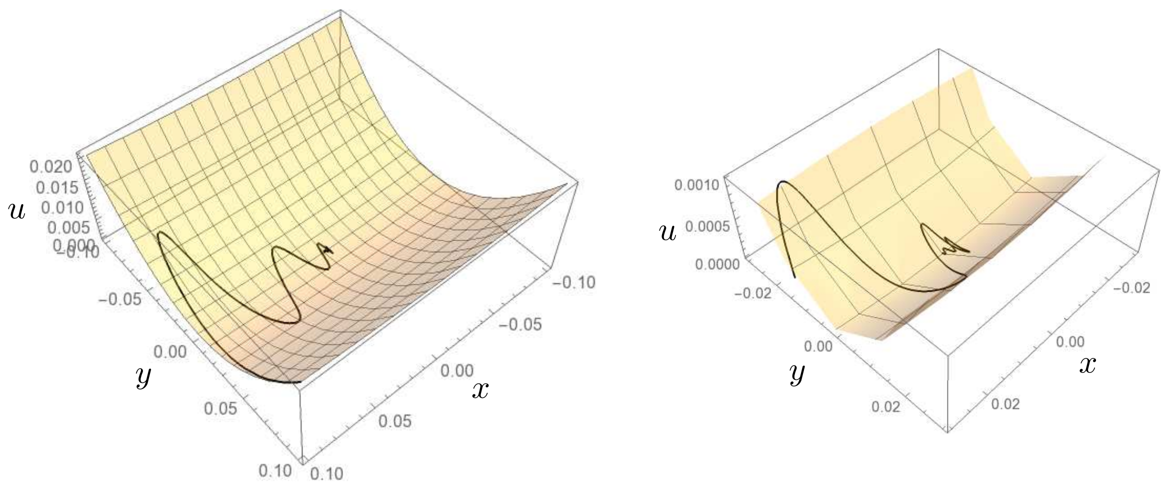


Figure 5: These figures show the path of motion on the graph of $u(x, y) = x^2/10 + 2y^2$ starting at $\mathbf{r}(0) = (x_0, y_0, u(x_0, y_0)) = (0.1, 0.1, 0.21)$ with initial velocity $\mathbf{r}'(0) = (x_0', y_0', z_0') = (0, -0.1, -0.04)$. The entire path is shown on the left and a portion corresponding to later times is shown on the right. Note that we have not followed the usual convention of labeling the axes along the positive ends, but rather the scale adjacent to the label represents measurement along the labeled axis.

decreasing in amplitude, in this case with respect to the y displacement, along with monotone decreasing values in the other direction, say the x-direction. The motion shown in Figure 5 seems to achieve this initially. Eventually however, the x values become negative (after several decaying oscillations in y values), and then the x displacement starts to increase back toward zero. I suspect there are orbits/motions with this property, but I have not found any nor do I have any systematic way to determine initial values for which this will be the case.