Math 2550, Final Exam PRACTICE Name and section: ________________

Note: This practice exam has (roughly) two problems from each chapter covered in the course (Chapters 12-15 of Thomas' Calculus). Each of the eight problems is scored for 10 points; you start with 20 points (free) for a total possible score of 100.

1. (10 points) (12.3-5) Let $\mathbf{u} = \mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{v} = -\mathbf{e}_1/2 + \sqrt{6}\,\mathbf{e}_2/4 + \sqrt{6}\,\mathbf{e}_3/4$.

   (a) Find $\mathbf{v} \cdot \mathbf{u}$.

   (b) Find $|\mathbf{v}|$.

   (c) Find the angle between $\mathbf{v}$ and $\mathbf{u}$.

   (d) Find the scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$.

   (e) Find the vector $\text{proj}_{\mathbf{u}}\,\mathbf{v}$.

   (f) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

   (g) Find a vector $\mathbf{w}$ for which $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are the vertices of an equilataral triangle.

   (h) Find the equation of the plane P determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

   (i) Find parametric equations for the line L through the origin and orthogonal to the plane P.

   (j) Eliminate the parameter in the parametric equations for L to obtain a system of equations for the line L.

Name and section: ____________________

Solution:

(a) $-1/2$

(b) 1

(c) $\cos\theta = -1/2$, so θ is an angle in the second quadrant with reference angle $\pi - \theta = \pi/3$. Therefore, $\theta = 2\pi/3$.

(d) $-1/2$

(e) $(\mathbf{v}\cdot\mathbf{u}/|\mathbf{u}|)\mathbf{u}/|\mathbf{u}| = (-1/2)\mathbf{u} = -\mathbf{e}_1/2 = (-1/2, 0, 0)$

(f) $|\mathbf{u}\times\mathbf{v}| = |-\mathbf{e}_1\times\mathbf{e}_1/2 + \sqrt{6}\,\mathbf{e}_1\times\mathbf{e}_2/4 + \sqrt{6}\,\mathbf{e}_1\times\mathbf{e}_3/4| = \sqrt{6}\,|\mathbf{e}_3 - \mathbf{e}_2|/4 = \sqrt{6}\,\sqrt{2}\,/4 = \sqrt{3}/2$.

(g) Notice from part (b) that $|\mathbf{u}| = |\mathbf{v}| = 1$ and that from part (c) that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $2\pi/3$. This means that the $\mathbf{u}$, $\mathbf{v}$, and the origin $\mathbf{o} = (0,0,0)$ form a triangle which is a part of an equilateral triangle with vertices $\mathbf{u}$ and $\mathbf{v}$ and center/centroid at the origin. Notice that the vertex $\mathbf{w}$ will lie in the plane $x = -1/2$ along with $\mathbf{v}$, and the projections of $\mathbf{v}$ and $\mathbf{w}$ into the y,z-plane will be symmetric with respect to the origin. That is,

$$\mathbf{w} = -\mathbf{e}_1/2 - \sqrt{6}\,\mathbf{e}_2/4 - \sqrt{6}\,\mathbf{e}_3/4.$$

This is probably the easiest way to find a vector $\mathbf{w}$ that will work. If one does not notice that the origin can be the center of the triangle, then you probably will have to work harder along the following lines:

There is a circle of vectors $\mathbf{w}$ that will work. The center of that circle is

$$\mathbf{p} = (\mathbf{u}+\mathbf{v})/2 = \mathbf{e}_1/4 + \sqrt{6}\,\mathbf{e}_2/8 + \sqrt{6}\,\mathbf{e}_3/8,$$

and the radius is the height of the equilateral triangle. Since the side of the equilateral triangle has length $|\mathbf{u}-\mathbf{v}| = \sqrt{9/4 + 3/4} = \sqrt{3}$, the height is $r = (\sqrt{3}/2)\sqrt{3} = 3/2$. The circle lies in the plane orthogonal to

$$\mathbf{u} - \mathbf{v} = 3\mathbf{e}_1/2 - \sqrt{6}\,\mathbf{e}_2/4 - \sqrt{6}\,\mathbf{e}_3/4$$

and through the point $\mathbf{p}$. So we need an orthonormal basis spanning the plane of the circle. That plane is

$$\{\mathbf{x} : (\mathbf{x}-\mathbf{p})\cdot(\mathbf{u}-\mathbf{v}) = 0\}.$$

This equation is

$$\mathbf{x}\cdot\left(\frac{3}{2}, -\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{4}\right) = \mathbf{p}\cdot(\mathbf{u}-\mathbf{v}) = \frac{3}{8} - \frac{3}{8} = 0.$$

Name and section: ____________________

The fact that $\mathbf{p} \cdot (\mathbf{u} - \mathbf{v})$ turns out to be zero should probably alert you to the fact that something is up. In particular, this means the origin is on a line which could contain the centroid. (In fact it could be the centroid.) But let's say we don't notice that. We're still in a position to look for vectors $\mathbf{x}$ having the property that $\mathbf{x} \cdot (\mathbf{u} - \mathbf{v}) = 0$. And we've just found one of them, namely, $\mathbf{p}$. Thus, we can take one vector in our orthonormal basis to be $\mathbf{a} = \mathbf{p}/|\mathbf{p}| = \mathbf{e}_1/2 + \sqrt{6}\,\mathbf{e}_2/4 + \sqrt{6}\,\mathbf{e}_3/4$ since $|\mathbf{p}| = 1/2$. Now, we can compute $(\mathbf{u} - \mathbf{v}) \times \mathbf{a}$ to get a second orthonormal basis vector:

$$\begin{aligned} 16(\mathbf{u} - \mathbf{v}) \times \mathbf{a} &= (6\mathbf{e}_1 - \sqrt{6}\,\mathbf{e}_2 - \sqrt{6}\,\mathbf{e}_3) \times (2\mathbf{e}_1 + \sqrt{6}\,\mathbf{e}_2 + \sqrt{6}\,\mathbf{e}_3) \\ &= 6\sqrt{6}\,\mathbf{e}_3 - 6\sqrt{6}\mathbf{e}_2 + 2\sqrt{6}\,\mathbf{e}_3 - 6\mathbf{e}_1 - 2\sqrt{6}\,\mathbf{e}_2 + 6\mathbf{e}_1 \\ &= -8\sqrt{6}\,\mathbf{e}_2 + 8\sqrt{6}\,\mathbf{e}_3. \end{aligned}$$

Thus, we may take $\mathbf{b} = (-\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}$. Therefore, every vector $\mathbf{w}$ that can be used has the form

$$\mathbf{w} = \mathbf{p} + (3/2)[(\cos\theta)\mathbf{a} + (\sin\theta)\mathbf{b}]. \tag{1}$$

The choice $\theta = \pi$ should give us the vertex for which the origin is the centroid, namely $\mathbf{p} - (3/2)\mathbf{a}$. And indeed, it's easy to check that this is the "easy" value of $\mathbf{w}$ we found above. But again, any of the vectors given in (1) can the the third vertex of an equilateral triangle along with $\mathbf{u}$ and $\mathbf{v}$. So, we'll give the rest of the answers in terms of an arbitrary value of θ and then specialize to $\theta = \pi$ for the "obvious" one.

(h) To get a plane, we need a normal. If we are able to visualize the situation geometrically and make the easiest choice of $\mathbf{w}$ in the previous part, then we have the origin $\mathbf{o}$ as centroid of the triangle, and we can take as normal $\mathbf{u} \times \mathbf{v} = \sqrt{6}(\mathbf{e}_3 - \mathbf{e}_2)/4$ which was computed in part (f). This gives the normal $\mathbf{e}_2 - \mathbf{e}_3$ and the plane through $\mathbf{e}_1$

$$(\mathbf{x} - \mathbf{e}_1) \cdot (\mathbf{e}_2 - \mathbf{e}_3) = 0 \qquad \text{or} \qquad y = z.$$

If we do not visualize the geometry, life becomes harder:

We know $\mathbf{u} - \mathbf{v}$ lies in the plane, so we can find a normal by crossing $\mathbf{u} - \mathbf{v}$ with

Name and section: ____________________

$\mathbf{w} - \mathbf{v}$:

$$
\begin{aligned}
(\mathbf{w}-\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) = \Bigg[& \left(\frac{\mathbf{e}_1}{4} + \frac{\sqrt{6}\,\mathbf{e}_2}{8} + \frac{\sqrt{6}\,\mathbf{e}_3}{8} \right) \\
& + \frac{3\cos\theta}{2} \left(\frac{\mathbf{e}_1}{2} + \frac{\sqrt{6}\,\mathbf{e}_2}{4} + \frac{\sqrt{6}\,\mathbf{e}_3}{4} \right) \\
& + \frac{3\sin\theta}{2} \left(-\frac{\mathbf{e}_2}{\sqrt{2}} + \frac{\mathbf{e}_3}{\sqrt{2}} \right) \\
& - \left(-\frac{\mathbf{e}_1}{2} + \frac{\sqrt{6}\,\mathbf{e}_2}{4} + \frac{\sqrt{6}\,\mathbf{e}_3}{4} \right) \Bigg] \\
& \times \left(\frac{3\mathbf{e}_1}{2} - \frac{\sqrt{6}\,\mathbf{e}_2}{4} - \frac{\sqrt{6}\,\mathbf{e}_3}{4} \right) \\
= \Bigg[& \left(\frac{3\mathbf{e}_1}{4} - \frac{\sqrt{6}\,\mathbf{e}_2}{8} - \frac{\sqrt{6}\,\mathbf{e}_3}{8} \right) \\
& + \frac{3\cos\theta}{2} \left(\frac{\mathbf{e}_1}{2} + \frac{\sqrt{6}\,\mathbf{e}_2}{4} + \frac{\sqrt{6}\,\mathbf{e}_3}{4} \right) \\
& + \frac{3\sin\theta}{2} \left(-\frac{\mathbf{e}_2}{\sqrt{2}} + \frac{\mathbf{e}_3}{\sqrt{2}} \right) \Bigg] \\
& \times \left(\frac{3\mathbf{e}_1}{2} - \frac{\sqrt{6}\,\mathbf{e}_2}{4} - \frac{\sqrt{6}\,\mathbf{e}_3}{4} \right) \\
= & \left(-\frac{3\sqrt{6}\,\mathbf{e}_3}{16} + \frac{3\sqrt{6}\,\mathbf{e}_2}{16} + \frac{3\sqrt{6}\,\mathbf{e}_3}{16} + \frac{3\mathbf{e}_1}{16} - \frac{3\sqrt{6}\,\mathbf{e}_2}{16} - \frac{3\mathbf{e}_1}{16} \right) \\
& + \frac{3\cos\theta}{2} \left(-\frac{\sqrt{6}\,\mathbf{e}_3}{8} + \frac{\sqrt{6}\,\mathbf{e}_2}{8} - \frac{3\sqrt{6}\,\mathbf{e}_3}{8} - \frac{3\mathbf{e}_1}{8} + \frac{3\sqrt{6}\,\mathbf{e}_2}{8} + \frac{3\mathbf{e}_1}{8} \right) \\
& + \frac{3\sin\theta}{2\sqrt{2}} \left(\frac{3\mathbf{e}_3}{2} + \frac{\sqrt{6}\,\mathbf{e}_1}{4} + \frac{3\mathbf{e}_2}{2} + \frac{\sqrt{6}\,\mathbf{e}_1}{4} \right) \\
= & \frac{3\sqrt{3}\,\sin\theta}{4}\,\mathbf{e}_1 \\
& + \left(\frac{3\sqrt{6}\,\cos\theta}{4} + \frac{9\sin\theta}{4\sqrt{2}} \right) \mathbf{e}_2 \\
& + \left(-\frac{3\sqrt{6}\,\cos\theta}{4} + \frac{9\sin\theta}{4\sqrt{2}} \right) \mathbf{e}_3.
\end{aligned}
$$

Thus, we can take as a normal

$$N = 3\sqrt{3}\,\sin\theta\,\mathbf{e}_1 + \left((3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)\mathbf{e}_2 + \left(-3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)\mathbf{e}_3,$$

and the equation of the plane P (noting that it passes through $\mathbf{u} = \mathbf{e}_1$ is

$$(\mathbf{x} - \mathbf{e}_1) \cdot N = 0$$

or

$$3\sqrt{3}\,\sin\theta\,x + \left((3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)y + \left(-3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)z = 3\sqrt{3}\,\sin\theta.$$

In the special case $\theta = \pi$, this simplifies to

$$y - z = 0$$

or simply $y = z$. Which can be easily checked for the simple choice of $\mathbf{w}$.

One may note that in the computation of the cross product above to get the normal, all the terms without a factor $\cos\theta$ or a factor $\sin\theta$ vanish. This should not be a surprise because this amounts to the assertion that $(\mathbf{p} - \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ vanishes, which is no surprise because $\mathbf{p}$ is the midpoint between $\mathbf{u}$ and $\mathbf{v}$, so obviously these two vectors are parallel.

(i) Since we have the normal N, the parametric equation for L is simply $\ell(t) = \mathbf{o} + tN = tN$. That is, in the general case

$$\begin{aligned}
x &= 3\sqrt{3}\,\sin\theta\,t \\
y &= \left(3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)t \\
z &= \left((-3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}\right)t.
\end{aligned}$$

When $\theta = \pi$, this can be simplified to

$$\begin{aligned}
x &= 0 \\
y &= t \\
z &= -t.
\end{aligned}$$

(j) By the first equation

$$t = \frac{1}{3\sqrt{3}\,\sin\theta}\,x$$

Name and section: ____________________

as long as θ is not an integer multiple of π. In this case, the line L may be expressed as the intersection of planes

$$y = \frac{3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}}{3\sqrt{3}\,\sin\theta}\,x \quad \text{and} \quad z = \frac{-3\sqrt{6}\,\cos\theta + \frac{9\sin\theta}{\sqrt{2}}}{3\sqrt{3}\,\sin\theta}\,x$$

or

$$2y = (2\sqrt{2}\,\cot\theta + \sqrt{6})\,x \quad \text{and} \quad 6z = (-2\sqrt{2}\,\cot\theta + \sqrt{6})\,x.$$

If $\sin\theta = 0$, then L may be written as the intersection of planes $x = 0$ and

$$y + z = 0.$$

This holds, in particular, in the simplest case $\theta = \pi$.

Remark: When I wrote this problem, I did not contemplate the possibility of a different choice of third vertex $\mathbf{w}$ and how difficult the last four parts would become if one did not see the obvious choice/answer. Perhaps the problem should be amended for the visually challenged to include a hint in part (g): You can take the origin as centroid.

Name and section: ____________________

2. (10 points) (12.6) Give a precise drawing of the surface associated with the relation within the square prism $[-1, 1] \times [-1, 1] \times \mathbb{R}$. (Make a reasonably good drawing, and label the boundary values, intercepts, etc..)

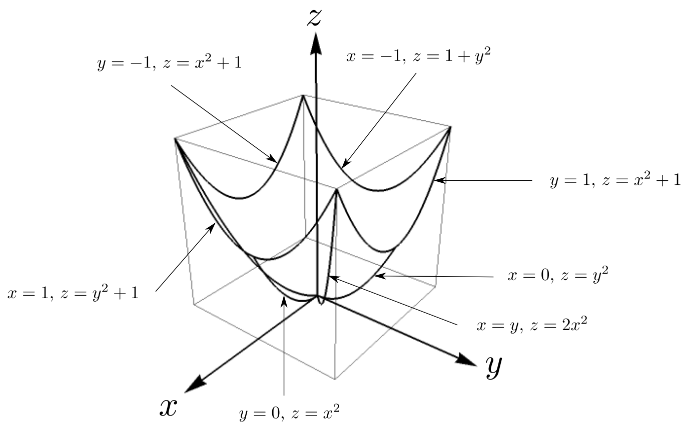
   (a) $z = x^2 + y^2$.

   (b) $z^2 = x^2 + y^2$

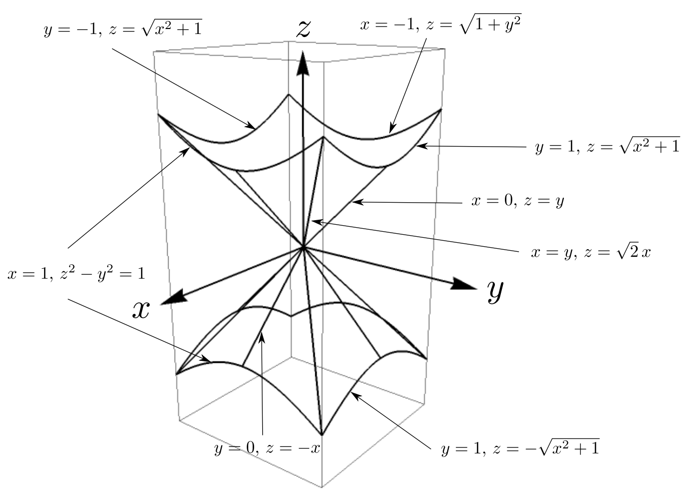
Name and section: ____________________

Solution:

(a) This is a paraboloid.



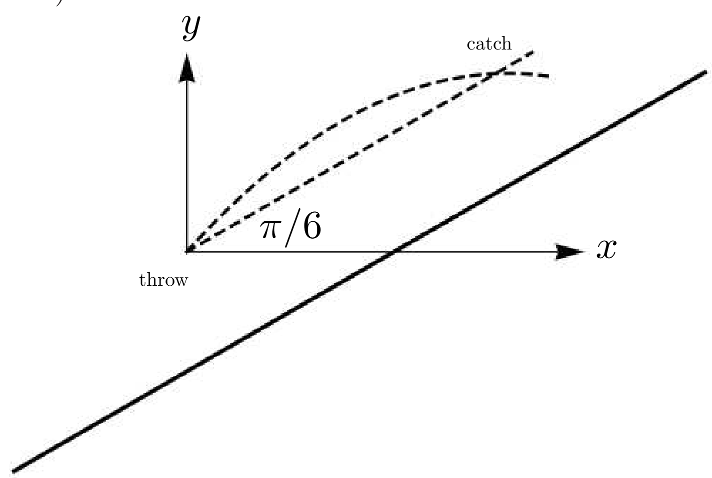
(b) This is a cone.



Name and section: ____________________

3. (10 points) (13.1-2) Two children play catch with a baseball on a 30 degree incline with one directly downhill from the other.

   (a) If w is the maxiumum velocity with which the ball can be thrown from the downhill side, what is the maximum distance up the hill (along the grade) the ball can be thrown. (You may assume either metric units with the acceleration due to gravity 9.8 m/s^2 or English measurement with $g = 32$ feet/s^2. You may assume the ball is caught at precisely the same distance from the ground (along grade) from which it is released/thrown.)



   (b) If W is the maximum velocity with which the ball can be thrown downhill, what is the maximum distance down the hill the ball can be thrown?

Name and section: ____________________

Solution:

(a) View the downhill side on the left and the uphill side on the right with x measuring horizontal distance. Assume the launch angle is $\theta > \pi/6$ measured from the horizontal x-axis. Then we have $\dot{x} = w\cos\theta$ and $\dot{y} = w\sin\theta - gt$. This means the path of the ball is parameterized as a function of time by

$$\mathbf{r}(t) = (wt\cos\theta, wt\sin\theta - gt^2/2).$$

We first seek the time, when the ball is caught. This is the first positive time t when

$$\frac{wt\sin\theta - gt^2/2}{wt\cos\theta} = \frac{w\sin\theta - gt/2}{w\cos\theta} = \tan(\pi/6) = 1/\sqrt{3}.$$

That is,

$$t = \frac{2w}{g}\left(\sin\theta - \frac{\cos\theta}{\sqrt{3}}\right).$$

Next, we find the angle θ for which $wt\cos\theta$ is maximum at this time. That is, we maximize the horizontal distance

$$d_1 = \frac{2w^2}{g}\cos\theta\left(\sin\theta - \frac{\cos\theta}{\sqrt{3}}\right)$$

as a function of θ. We can ignore the constant factor $2w^2/g$ for the moment, differentiate with respect to θ, and attempt to solve

$$-\sin\theta\left(\sin\theta - \frac{\cos\theta}{\sqrt{3}}\right) + \cos\theta\left(\cos\theta + \frac{\sin\theta}{\sqrt{3}}\right) = \cos^2\theta - \sin^2\theta + \frac{1}{\sqrt{3}}\sin(2\theta) = 0.$$

That is, $\cos(2\theta) + \sin(2\theta)/\sqrt{3} = 0$, or

$$\tan(2\theta) = -\sqrt{3}.$$

We conclude the optimal angle for the throw (from the horizontal) satisfies $2\theta = 2\pi/3$. Therefore, the best angle for the throw is $\pi/3$, or 60 degrees, precisely twice the angle of the slope and the angle bisecting the angle between the slope and the vertical for the downhill player. (One of these characterizations always holds no matter what the angle of the hill—can you guess which one? Can you prove it?)

Now we can compute directly $d_1/\cos(\pi/6) = 2d_1/\sqrt{3}$ with $\theta = \pi/3$. It may be helpful to observe that d_1 may be expressed in terms of 2θ as

$$d_1 = \frac{w^2}{g}\left(\sin(2\theta) - \frac{\cos(2\theta) + 1}{\sqrt{3}}\right).$$

Either way, the horizontal distance with $\theta = \pi/3$ is $d_1 = w^2(\sqrt{3}/2 - \sqrt{3}/6)/g$, so the maximum distance along grade is

$$\frac{2d_1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\frac{w^2\sqrt{3}}{3g}\right) = \frac{2w^2}{3g}.$$

Name and section: ____________________

(b) We repeat the procedure above looking from the other side, so the positive x-axis points in the downhill direction:

Now the launch angle is $\theta > -\pi/6$. The equation for the time of flight is

$$\frac{Wt\sin\theta - gt^2/2}{wt\cos\theta} = \frac{w\sin\theta - gt/2}{W\cos\theta} = \tan(-\pi/6) = -1/\sqrt{3}$$

with solution

$$t = \frac{2W}{g}\left(\sin\theta + \frac{\cos\theta}{\sqrt{3}}\right).$$

Differentiating as before (taking account of the sign change in the expression for the maximum horizontal distance d_2, we find the optimal angle satisfies $\cos(2\theta) - \sin(2\theta)/\sqrt{3} = 0$ or $\tan(2\theta) = \sqrt{3}$. This means $\theta = \pi/6$. Plugging back into the expression for d_2, we get

$$d_2 = \frac{2W^2}{g}\cos\theta\left(\sin\theta + \frac{\cos\theta}{\sqrt{3}}\right) = \frac{W^2}{g}\left(\sin(2\theta) + \frac{\cos(2\theta)+1}{\sqrt{3}}\right) = \frac{W^2\sqrt{3}}{g}.$$

Hence, the max downhill throw has distance along grade

$$\frac{d_2}{\cos(\pi/6)} = \frac{2d_2}{\sqrt{3}} = \frac{2W^2}{g}.$$

The uphill player (of equal arm strength/initial throwing velocity $W = w$) can throw three times farther. In order to have a good match for playing catch on this hill, perhaps the uphill thrower should have a stronger throw so that $w = W\sqrt{3}$.

Name and section: ____________________

4. (10 points) (13.3-4) Let $\mathbf{r}(t) = (3\sin t, 3\cos t, 4t)$ parameterize a curve in $\mathbb{R}^3$.

   (a) Reparameterize the curve by arclength.

   (b) Find the curvature vector at each point on the image curve.

Name and section: ____________________

Solution:

(a) $\mathbf{r}' = (3\cos t, -3\sin t, 4)$ and $|\mathbf{r}| = 5$, so $s = 5t$, and a reparameterization by arclength is

$$\gamma(s) = \left(3\sin\frac{s}{5}, 3\cos\frac{s}{5}, \frac{4s}{5}\right).$$

(b)

$$\gamma' = \frac{1}{5}\left(3\cos\frac{s}{5}, -3\sin\frac{s}{5}, 4\right)$$

and the curvature vector is

$$\gamma'' = -\frac{3}{25}\left(\sin\frac{s}{5}, \cos\frac{s}{5}, 0\right).$$

Name and section: ____________________

5. (10 points) (14.1-3) Verify that the function

$$u(x, y, z) = e^{3x+4y} \cos(5z)$$

satisfies Laplace's partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Name and section: ____________________

Solution:

$$\frac{\partial u}{\partial x} = 3e^{3x+4y}\cos(5z), \qquad \frac{\partial u}{\partial y} = 4e^{3x+4y}\cos(5z), \qquad \text{and} \qquad \frac{\partial u}{\partial z} = -5e^{3x+4y}\sin(5z).$$

$$\frac{\partial^2 u}{\partial x^2} = 9e^{3x+4y}\cos(5z), \qquad \frac{\partial^2 u}{\partial y^2} = 16e^{3x+4y}\cos(5z), \qquad \text{and} \qquad \frac{\partial^2 u}{\partial z^2} = -25e^{3x+4y}\cos(5z).$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (9+16-25)e^{3x+4y}\cos(5z) = 0.$$

Name and section: ____________________

6. (10 points) (14.1-8) Identify the maximum and minimum points for the function $f(x, y) = xy$ on the circle $x^2 + y^2 = 10$.

Name and section: ____________________

Solution: Let's use the method of Lagrange multipliers. Setting $g(x,y) = x^2+y^2$ we note that $\nabla g = (2x, 2y) \neq (0,0)$ on the circle. Furthermore, $\nabla f = (y, x)$, so we look for a constant λ and a point (x, y) for which $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $x^2+y^2 = 10$. The gradient conditions give

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y. \end{cases}$$

Noting that it is not possible to have a solution of this system with either $x = 0$ or $y = 0$, we can eliminate λ by dividing the last two equations:

$$\frac{y}{x} = \frac{x}{y}.$$

From this, we conclude $x^2 = y^2$. Thus, there are four such points on the circle represented by $(\pm\sqrt{5}, \pm\sqrt{5})$. If x and y have the same sign, we can take $\lambda = 1/2$ and get a full solution, and if x and y have opposite signs, we can take $\lambda = -1/2$ and get a solution. Thus, all four points are solutions of this system. In the former case, $f(x,y) = 5$ corresponding to two

$$\text{maximum points:} \qquad (\sqrt{5}, \sqrt{5}) \quad \text{and} \quad (-\sqrt{5}, -\sqrt{5}).$$

If x and y have opposite signs, we get two

$$\text{minimum points:} \qquad (\sqrt{5}, -\sqrt{5}) \quad \text{and} \quad (-\sqrt{5}, \sqrt{5}).$$

Name and section: ____________________

7. (15.1-4) Evaluate the iterated integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0}\frac{2}{1+\sqrt{x^2+y^2}}\,dydx.$$

Name and section: ____________________

Solution: This is integration over the quarter disk in the third quadrant. The result should be the same as integration over the quarter disk in the first quadrant, but we'll work make both calculations (using polar coordinates in both of them). Over the given quarter disk, we get

$$\begin{aligned}\int_0^1 \int_\pi^{3\pi/2} \frac{2r}{1+r}\, d\theta dr &= \frac{\pi}{2} \int_0^1 \frac{2r}{1+r}\, dr \\ &= \pi \int_0^1 \left(1 - \frac{1}{1+r}\right) dr \\ &= \pi \left(1 - \ln 2\right).\end{aligned}$$

Since there is no θ dependence in the integrand, the integral over the quarter disk in the first quadrant is essentially the same:

$$\begin{aligned}\int_0^1 \int_0^{\pi/2} \frac{2r}{1+r}\, d\theta dr &= \frac{\pi}{2} \int_0^1 \frac{2r}{1+r}\, dr \\ &= \pi \int_0^1 \left(1 - \frac{1}{1+r}\right) dr \\ &= \pi \left(1 - \ln 2\right).\end{aligned}$$

Name and section: ____________________

8. (10 points) (15.1-8) A plate is modeled by the region

$$\mathcal{U} = \{(x, y, 0) \in \mathbb{R}^3 : b^2x^2 + a^2y^2 < a^2b^2\}$$

along with an areal density $\delta = 1 + bx^2 + a^2y^2$ in the given coordinates where a and b are positive constants. If one models uniform rotation of this plate about the z-axis with angular velocity $\omega = 5$ radians per second, calculate the kinetic energy of rotation. Hint: Use the relations $x = ra\cos\theta$ and $y = rb\sin\theta$ to change variables.

Name and section: ____________________

Solution: First of all, the change of variables reparameterizes the plate on the unit disk in polar coordinates which is a rectangle

$$\mathcal{R} = [0,1) \times [0, 2Pi) = \{(r,\theta) : 0 \le r < 1,\ 0 \le \theta < 2\pi\}.$$

We can go ahead and calculate the scaling factor:

$$\sigma = \begin{vmatrix} a\cos\theta & -ra\sin\theta \\ b\sin\theta & rb\cos\theta \end{vmatrix} = abr.$$

The kinetic energy is $I\omega^2/2 = 25I/2$ where I is the moment of inertia.

$$\begin{aligned} I &= \int_{\mathcal{U}} \delta r^2 \\ &= \int_{\mathcal{R}} (1 + a^2b^2r^2)r^2\sigma \\ &= \int_0^1 \int_0^{2\pi} (1 + a^2b^2r^2)r^2 abr\, d\theta dr \\ &= 2ab\pi \int_0^1 (r^3 + a^2b^2r^5)\, dr \\ &= 2ab\pi(1/4 + a^2b^2/6) \\ &= ab\pi(1/2 + a^2b^2/3). \end{aligned}$$

Thus, the kinetic energy is

$$\text{K.E.} = \frac{25ab\pi}{6}(3 + 2a^2b^2).$$