Math 2550, Final Exam PRACTICE Name and section: ____________________

Note: This practice exam has (roughly) two problems from each chapter covered in the course (Chapters 12-15 of Thomas' Calculus). Each of the eight problems is scored for 10 points; you start with 20 points (free) for a total possible score of 100.

1. (10 points) (12.3-4, 13.1) A point mass m moves around a circle (circular motion) with position given by

$$\mathbf{r}(t) = r(\cos\theta, \sin\theta, 0).$$

where $r > 0$ is constant but θ depends on time t.

(a) Recall Newton's second law which states that $\mathbf{f} = m\mathbf{a}$ where $\mathbf{f}$ is the total force on the point mass and $\mathbf{a}$ is the acceleration. Derive an expression (i.e., find a formula) for the **torque vector** $\vec{\tau} = \mathbf{r} \times \mathbf{f}$.

(b) If we let $\omega = \omega(t) = \theta'(t)$ denote the angular velocity of the point mass, then identify the quantity I for which Newton's second law may be written as

$$\vec{\tau} = I\omega'\mathbf{e}_3 = I\omega'(0, 0, 1).$$

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(a) $\mathbf{v} = \mathbf{r}' = r\theta'(-\sin\theta, \cos\theta, 0)$ and

$$\mathbf{a} = \mathbf{r}'' = r[\theta''(-\sin\theta, \cos\theta, 0) - (\theta')^2(\cos\theta, \sin\theta, 0)].$$

Therefore

$$\begin{aligned}\vec{\tau} &= r(\cos\theta, \sin\theta, 0) \times mr[\theta''(-\sin\theta, \cos\theta, 0) - (\theta')^2(\cos\theta, \sin\theta, 0)] \\ &= mr^2\theta'' \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \\ &= mr^2\theta'' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

(b) If $\omega = \theta'$, then $\theta'' = \omega'$, so Newton's second law reads

$$\vec{\tau} = mr^2\omega'\mathbf{e}_3.$$

This means we should take $I = mr^2$, the second moment of the mass with respect to the z-axis. Notice that this formula gives a parallel version of Newton's second law $\mathbf{f} = m\mathbf{a}$ for rotational motion with torque replacing force and angular velocity replacing linear velocity $\mathbf{v} = \mathbf{r}'$. In particular, we can write

$$\begin{aligned}|\mathbf{f}| &= m|\mathbf{v}'| \\ |\vec{\tau}| &= I|\omega'|.\end{aligned}$$

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2. (10 points) (12.6) Give a precise drawing of the surface associated with the relation within the square prism $[-1, 1] \times [-1, 1] \times \mathbb{R}$. (Make a reasonably good drawing, and label the boundary values, intercepts, etc..)

   (a) $x^2 + z^2 = 1$.

   (b) $x^2 + z = y^2$

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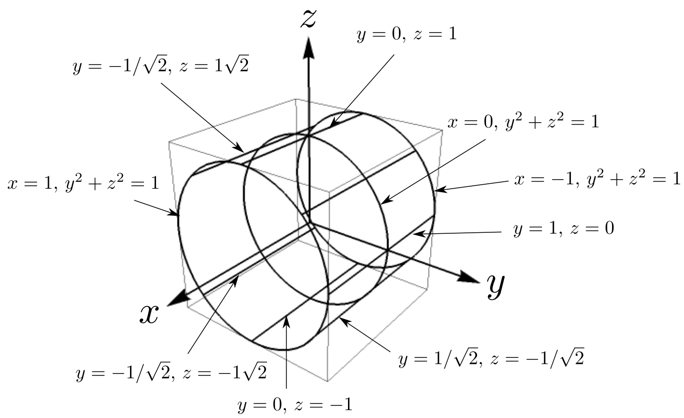
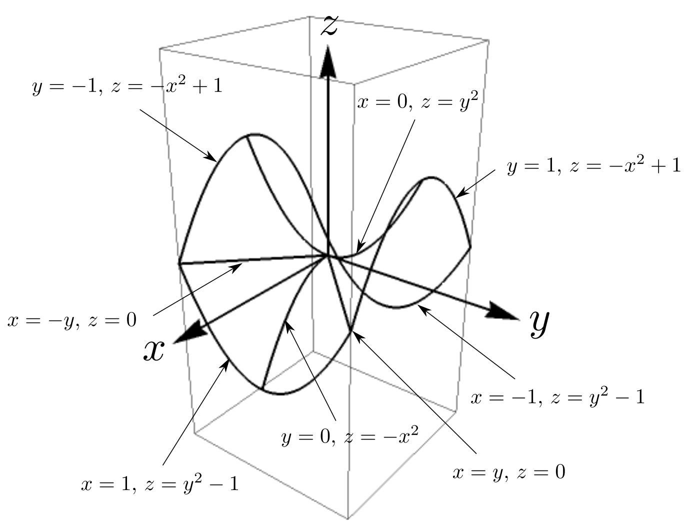
Solution:

(a) This is a cylinder.



(b) This is a (hyperbolic) paraboloid.



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3. (10 points) (12.6, 13.1) Find a parameterization of the curve in $\mathbb{R}^3$ determined by the system of equations

$$\begin{cases} z + x^2 = y^2, \\ x^2 + y^2 = 1, \end{cases}$$

which is the intersection of the surface $z + x^2 = y^2$ from the previous problem with a cylinder.

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Solution: The circle which generates the cylinder may be parameterized with $x = \cos t$ and $y = \sin t$. Therefore, the intersection curve may be parameterized by

$$\mathbf{r}(t) = (\cos t, \sin t, -\cos^2 t + \sin^2 t) = (\cos t, \sin t, -\cos 2t).$$

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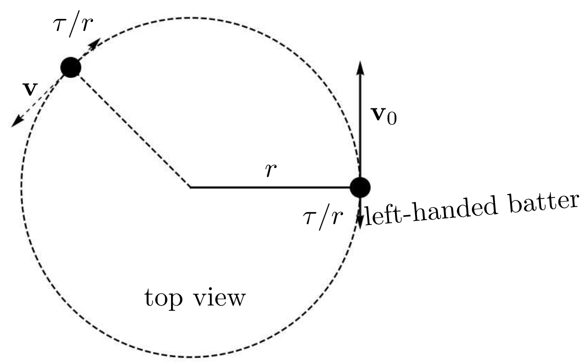
4. (10 points) (13.1-4) A swing trainer for practicing batting in baseball is constructed as shown from above in the diagram with a ball swinging in a circle in a horizontal plane. We assume the impact of the bat with the ball (in the position closest to the batter) imparts an instantaneous velocity $\mathbf{v}_0 = v_0(0, 1)$.



Assume the following:

1. The arm is weightless (and has length r).
2. The ball has mass m.
3. The arm imparts a uni-directional constant torque τ. This means there is a force of **constant magnitude** τ/r in the direction **opposite to the forward circular motion** of the ball.

Calculate the maximum velocity for which the ball will not go beyond the position farthest from the batter.

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Solution: The motion of the ball is given by $\mathbf{r}(t) = r(\cos\theta, \sin\theta)$ where $\theta = \theta(t)$ satisfies $\theta(0) = 0$ and is increasing. The velocity is given by

$$\mathbf{v} = \mathbf{r}' = r\theta'(-\sin\theta, \cos\theta)$$

and the **forward circular motion** is in the direction $(-\sin\theta, \cos\theta)$. Therefore, we know $\mathbf{r}'(0) = r\theta'(0)(0,1)$, and

$$\theta'(0) = v_0/r. \tag{1}$$

The acceleration is

$$\mathbf{r}'' = r[\theta''(-\sin\theta, \cos\theta) - \theta'^2(\cos\theta, \sin\theta)].$$

Therefore by Newton's second law

$$mr[\theta''(-\sin\theta, \cos\theta) - \theta'^2(\cos\theta, \sin\theta)] = -T(\cos\theta, \sin\theta) - (\tau/r)(-\sin\theta, \cos\theta) \tag{2}$$

where $-T(\cos\theta, \sin\theta)$ is the tension force which adjusts to prevent any motion outside the circle.

Notice the **units** of the torque: $[\tau] = [\mathbf{r} \times \mathbf{f}] = \mathrm{LM}(\mathrm{L}/\mathrm{T}^2) = \mathrm{ML}^2/\mathrm{T}^2$. See the first problem on this exam.

Notice that the direction normal to the motion $\mathbf{u}_1 = (\cos\theta, \sin\theta)$ and and the direction of the motion $\mathbf{u}_2 = (-\sin\theta, \cos\theta)$ constitute an orthonormal basis.

Equating the components of $(-\sin\theta, \cos\theta)$ in (2), we find

$$mr\theta'' = -\frac{\tau}{r}.$$

Therefore,

$$\theta'' = -\frac{\tau}{mr^2}. \tag{3}$$

Again, this is compatible with the equation $\vec{\tau} = mr^2\theta''(0,0,1) = I\omega'\mathbf{e}_3$ of the first problem and the units $[-\tau/(mr^2)] = \mathrm{ML}^2/(\mathrm{T}^2\mathrm{ML}^2) = 1/\mathrm{T}^2$ are dimensionally correct.

Integrating (3) a first time and using (1), we get

$$\theta'(t) = \theta'(0) - \frac{\tau}{mr^2}\,t = \frac{v_0}{r} - \frac{\tau}{mr^2}\,t.$$

This means that the forward motion of the ball ceases when $t = mrv_0/\tau$. Again, we can check the units: $[mrv_0/\tau] = (\mathrm{ML}^2/\mathrm{T})/[\tau] = (\mathrm{ML}^2/\mathrm{T})/(\mathrm{ML}^2/\mathrm{T}^2) = T$.

Integrating a second time from $t = 0$ where $\theta(0) = 0$, we find

$$\theta(t) = \frac{v_0}{r}\,t - \frac{\tau}{2mr^2}\,t^2.$$

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Therefore evaluating at the first stopping time $t_{\text{stop}} = mrv_0/\tau$ we get the desired limiting equation

$$\theta(t_{\text{stop}}) = \frac{mv_0^2}{\tau} - \frac{1}{2}\,\frac{mv_0^2}{\tau} \leq \pi.$$

($\theta = \pi$ gives the position farthest from the batter.) This means

$$v_0 \leq \sqrt{\frac{2\pi\tau}{m}}.$$

Notice that if the opposing torque goes up or the mass goes down, then the maximum initial velocity can go up.

Note: The problem could be made more realistic if a restoring force/spring were included to bring the ball back to the initial position for the next swing of the bat. This force might be considered in addition to the constant torque above or might replace the constant torque ($\tau = 0$). A simple term in Newton's second law to model such a restoring force would have the form $-cr|\theta|(-\sin\theta, \cos\theta)$ where c is some constant with units $[c] = \mathrm{M}/\mathrm{T}^2$. To solve such a problem, you would need to solve the second order linear equation $mr\theta'' = -cr\theta - \tau/r$. This is a relatively easy ordinary differential equation covered in any elementary course on differential equations.

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5. (10 points) (14.1-4) $u(x, y, z) = e^{qr} \sin^{-1} p$ where

$$\begin{cases} p(x) = \sin x, \\ q(y, z) = z^2 \ln y, \\ r(z) = 1/z. \end{cases}$$

Use the chain rule to find the first partials of u with respect to x, y, and z evaluated at the point $(x, y, z) = (\pi/4, 1/2, -1/2)$.

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Solution:
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\,\frac{\partial p}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}\ \cos x,$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q}\,\frac{\partial q}{\partial y} = re^{qr}\sin^{-1} p\ \frac{z^2}{y},$$
and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}\,\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial z} = re^{qr}\sin^{-1} p\ 2z\ln y + qe^{qr}\sin^{-1} p\ \left(-\frac{1}{z^2}\right).$$
At $(x, y, z) = (\pi/4, 1/2, -1/2)$ we have $p = \sqrt{2}/2$, $q = (-\ln 2)/4$, and $r = -2$. Therefore, $qr = \ln\sqrt{2}$, and
$$\frac{\partial u}{\partial x} = \frac{\sqrt{2}}{\sqrt{1/2}}\ \frac{\sqrt{2}}{2} = \sqrt{2},$$
$$\frac{\partial u}{\partial y} = -2\sqrt{2}\ \sin^{-1}(\sqrt{2}/2)\ \frac{1}{2} = -\frac{\pi\sqrt{2}}{4},$$
and
$$\begin{aligned}\frac{\partial u}{\partial z} &= -2\sqrt{2}\ \sin^{-1}(\sqrt{2}/2)\ [-\ln(1/2)] + [(-\ln 2)/4]\sqrt{2}\ \sin^{-1}(\sqrt{2}/2)\ (-4)\\ &= -\frac{\pi\ln 2\sqrt{2}}{2} + \frac{\pi\ln 2\sqrt{2}}{4}\\ &= -\frac{\pi\ln 2\sqrt{2}}{4}.\end{aligned}$$

As a (double) check we can plug in the expressions for p, q and r to get $u(x, y, z)$ directly:
$$u(x, y, z) = xy^z.$$
Clearly, $\partial u/\partial x = y^z$. Evaluated at the point we get $(1/2)^{-(1/2)} = \sqrt{2}$. So that's correct. For the next partial,
$$\frac{\partial u}{\partial y} = xzy^{z-1}.$$
Evaluated at the point,
$$\frac{\partial u}{\partial y} = \frac{\pi}{4}\ \left(-\frac{1}{2}\right)(1/2)^{-3/2} = -\frac{\pi}{8}\ (2)^{-3/2} = -\frac{\pi}{4}\ \sqrt{2}.$$
This matches was we got above. Finally, by direct calculation
$$\frac{\partial u}{\partial z} = xy^z\ln y.$$

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Remember this one comes from writing u as

$$u(x,y,z) = xe^{z \ln y}$$

(and then differentiating with respect to z). In any case, when we evaluate with $x = \pi/4$, $y = 1/2$ and $z = -1/2$, we get

$$\frac{\partial u}{\partial z} = \frac{\pi}{4}\ \sqrt{2}\ (-\ln 2) = -\frac{\pi\sqrt{2}\ \ln 2}{4}.$$

Again, this is the same thing we got using the chain rule.

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6. (10 points) (14.1-7) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = (4x - x^2)\cos y$.

   (a) Find all (interior) critical points of f in $\mathbb{R}^2$ and classify them using the second derivative test.

   (b) Restrict the function to the rectangle

$$\mathcal{R} = [1, 3] \times [-\pi/4, \pi/4] = \{(x, y) : 1 \le x \le 3,\ |y| \le \pi/4\}$$

   and find the absolute minimum value.

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Solution: This is from problem 14.7.37.

(a) $\nabla f = ((4-2x)\cos y, -(4x-x^2)\sin y)$ and

$$D^2 f = \begin{pmatrix} -2\cos y & -(4-2x)\sin y \\ -(4-2x)\sin y & (x^2-4x)\cos y \end{pmatrix}.$$

Critical points (x,y) solve the system $(4-2x)\cos y = 0$ and $(x^2-4x)\sin y = 0$. From the first equation either $x = 2$ or $y = \pi/2 + k\pi$ for some $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. From the second equation $x = 0$, $x = 4$, or $y = k\pi$ for some $k \in \mathbb{Z}$. Thus, we seem to have three **kinds** of critical points:

1. $(2, k\pi)$
   In this case,
   $$D^2 f(2, k\pi) = \begin{pmatrix} -2(-1)^k & 0 \\ 0 & -4(-1)^k \end{pmatrix} = \begin{pmatrix} 2(-1)^{k+1} & 0 \\ 0 & 4(-1)^{k+1} \end{pmatrix}.$$
   Thus, there are really two cases here: If k is even, then both eigenvalues of $D^2 f$ are negative and there is a **local max** at $(2, k\pi)$ with value $f(2, k\pi) = 4$.
   If k is odd, then both eigenvalues are positive and there is a **local min** at $(2, k\pi)$ with value $f(2, k\pi) = -4$.
2. $(0, \pi/2 + k\pi)$
   In this case,
   $$D^2 f(0, \pi/2 + k\pi) = \begin{pmatrix} 0 & -4(-1)^k \\ -4(-1)^k & 0 \end{pmatrix}.$$
   Therefore $\det D^2 f < 0$, and these are **saddle points** with value $f(0, \pi/2 + k\pi) = 0$.
3. $(4, \pi/2 + k\pi)$
   In this case,
   $$D^2 f(0, \pi/2 + k\pi) = \begin{pmatrix} 0 & 4(-1)^k \\ 4(-1)^k & 0 \end{pmatrix}.$$
   Again $\det D^2 f < 0$, and these are **saddle points**. The value at these points is also $f(4, \pi/2 + k\pi) = 0$.

(b) It will first be noted that there is only one interior critical point $(2, 0) \in \mathbb{R}$, and this is a maximum. Thus, the minimum will be found on the boundary

$$\begin{aligned} \partial\mathcal{R} = &\{(1, y) : |y| \le \pi/4\} \cup \{(x, \pi/4) : 1 \le x \le 3\} \\ &\cup \{(3, y) : |y| \le \pi/4\} \cup \{(x, -\pi/4) : 1 \le x \le 3\}. \end{aligned}$$

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The values on the left side of the rectangle are $f(1, y) = 3\cos y$. The minimum values are at the corners with $f(1, \pm\pi/4) = 3/\sqrt{2}$.

Along the top $f(x, \pi/4) = (4x - x^2)/\sqrt{2}$. Again, the local min values are at the corners: $f(1, \pi/4) = 3/\sqrt{2}$ and $f(3, \pi/4) = 3/\sqrt{2}$. We know the min values are at the corners because $4x - x^2 = x(4 - x)$ has graph a parabola opening down and has zeros at $x = 0$ and $x = 4$. Since these ordinates are symmetrically placed $0 < 1 < 3 < 4$ with respect to the interval $[1, 3]$, we can also see by symmetry that the values at the corners are (again) the same.

Along the right side we get $f(3, y) = 3\cos y$, so this is like the left side.

Similarly along the bottom, $f(x, -\pi/4) = (4x - x^2)/\sqrt{2}$.

Conclusion: The absolute min value is $3/\sqrt{2} = 3\sqrt{2}/2$ and is taken in all four corners $(1, \pm\pi/4)$, $(3, \pm\pi/4)$ of $\mathcal{R}$.

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7. (10 points) (15.1-6) A plate is modeled by the region

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \leq 1\}$$

along with an areal density $\delta = x + y + x^2 + (y-1)^2$ in the given coordinates.

(a) Prove that the density is positive on $\mathcal{U}$.

(b) Find the center of mass of the plate.

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Solution:

(a) Looking at the expression for δ, we should complete the squares with respect to x and y. This will require expanding $(y-1)^2$ to absorb the additional y:

$$\begin{aligned}
\delta &= x + y + x^2 + (y-1)^2 \\
&= (x+1/2)^2 - 1/4 + y^2 - y + 1 \\
&= (x+1/2)^2 + (y-1/2) - 1/4 - 1/4 + 1 \\
&= (x+1/2)^2 + (y-1/2) + 1/2 \\
&\geq 1/2.
\end{aligned}$$

(b) At least part of the denstity $x^2 + (y-1)^2$ is symmetric with respect to the disk $\mathcal{U}$, so this (part) should leave the center of mass in the center of the disk.

Let's make a preliminary change of variables to shift the center to the origin and then use standard polar coordinates. The shift to the unit disk $B = \{(\xi, \eta) : \xi^2 + \eta^2 \leq 1\}$ is accomplished by $\Psi : B \to \mathcal{U}$ by $\Psi(\xi, \eta) = (\xi, \eta + 1)$ for which the scaling factor is clearly $\sigma = 1$. Thus,

$$\begin{aligned}
\int_{\mathcal{U}} x\delta &= \int_B \xi(\xi + \eta + 1 + \xi^2 + \eta^2) \\
&= \int_0^1 \int_0^{2\pi} r\cos\theta(r\cos\theta + r\sin\theta + 1 + r^2) r \, d\theta dr \\
&= \int_0^1 \int_0^{2\pi} [r^3 \cos^2\theta + r^3 \cos\theta \sin\theta + r^2(1+r^2)\cos\theta] \, d\theta dr \\
&= \int_0^1 r^3 \int_0^{2\pi} \cos^2\theta \, d\theta dr + \int_0^1 r^3 \int_0^{2\pi} \cos\theta \sin\theta \, d\theta dr + \int_0^1 r^2(1+r^2) \int_0^{2\pi} \cos\theta \, d\theta dr \\
&= (1/4) \int_0^{2\pi} \cos^2\theta \, d\theta \\
&= \pi/4.
\end{aligned}$$

Here we have used the "trick" that the integral of $\cos^2\theta$ over one period is half the integral of the constant 1 over the same period.

$$\begin{aligned}
\int_{\mathcal{U}} y\delta &= \int_B (\eta+1)(\xi+\eta+1+\xi^2+\eta^2) \\
&= \int_0^1 \int_0^{2\pi} (r\sin\theta+1)(r\cos\theta+r\sin\theta+1+r^2)r\,d\theta dr \\
&= \int_0^1 \int_0^{2\pi} \left[r^3\sin^2\theta + r + r^3\right] d\theta dr \\
&= \pi/4 + 2\pi \int_0^1 (r+r^3)\,dr \\
&= \pi/4 + 2\pi(1/2+1/4) \\
&= 7\pi/4.
\end{aligned}$$

$$\begin{aligned}
m &= \int_{\mathcal{U}} \delta \\
&= \int_B (\xi+\eta+1+\xi^2+\eta^2) \\
&= \int_0^1 \int_0^{2\pi} (r\cos\theta+r\sin\theta+1+r^2)r\,d\theta dr \\
&= 2\pi \int_0^1 \left[r+r^3\right] dr \\
&= 2\pi(1/2+1/4) \\
&= 3\pi/2.
\end{aligned}$$

Therefore, the center of mass of the shifted disk is

$$(\bar{\xi},\bar{\eta}) = ((\pi/4)/(3\pi/2), (7\pi/4)/(3\pi/2)) = (1/6, 7/6),$$

and the center of mass of the original disk $\mathcal{U}$ is

$$(\bar{x},\bar{y}) = (\bar{\xi},\bar{\eta}+1) = (1/6, 13/6).$$

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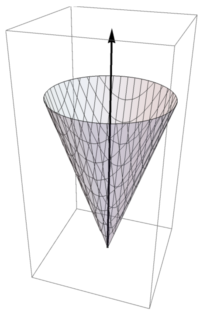
8. (10 points) (15.1-7) The solid circular cone shown below has radius $r = 1$, height $h = 3$, and constant density $\delta = 1$.



(a) Set up the integral in **cylindrical coordinates** for the second moment (moment of inertia) for this solid cone about the z-axis.

(b) Evaluate the integral

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Solution: This is (sort of) 15.7.29.

(a)

$$\int_0^{2\pi} \int_0^1 \int_{3r}^3 r^2\, r\, dz dr d\theta = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3\, dr dz d\theta.$$

(b)

$$\begin{aligned} I &= 2\pi \int_0^3 \frac{1}{4} \left(\frac{z}{3}\right)^4 dz \\ &= \frac{\pi}{2(3)^4} \frac{1}{5} (3)^5 \\ &= \frac{3\pi}{10}. \end{aligned}$$