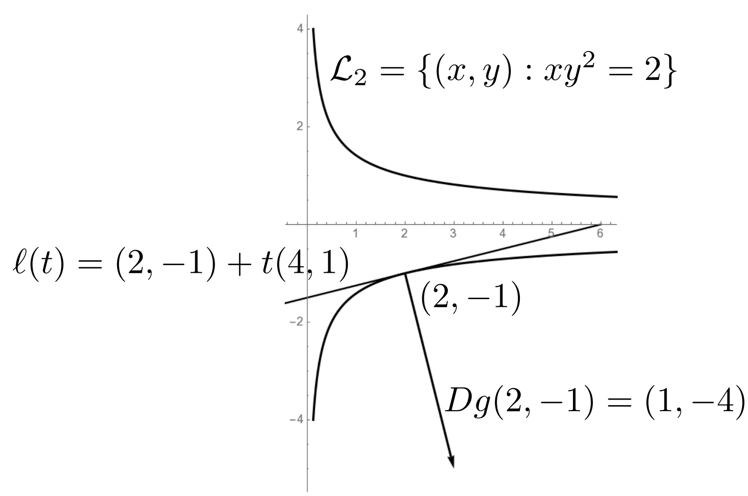
Math 2550, Exam 2 practice Name and section: ____________

1. (14.5.3) Let $g(x, y) = xy^2$.
   (a) (10 points) Draw the level curve of g passing through $(x, y) = (2, -1)$.

   (b) (5 points) Draw and label the gradient of g at the point $(x, y) = (2, -1)$ (on your drawing above).
   (c) (5 points) Draw the tangent line to the level curve at the point $(x, y) = (2, -1)$ (on your drawing above).

Solution:

(a) $g(2, -1) = 2$, so the level curve is $x = 2/y^2$.



(b) $Dg = (y^2, 2xy)$ and $Dg(2, -1) = (1, -4)$.

(c) The line $\ell(t) = (2, -1) + t(4, 1)$ is orthogonal to the gradient.

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2. (20 points) (14.5.17) Find the rate of change of the function $f(x, y, z) = 3e^x \cos(yz)$ in the direction $\mathbf{v} = (2, 1, -2)$ at the origin.

Solution: $\mathbf{u} = (2, 1, -2)/3$.

$$Df = (3e^x \cos(yz), -3ze^x \sin(yz), -3ye^x \sin(yz)).$$

$$Df(0, 0, 0) = (3, 0, 0).$$

$$D_{\mathbf{u}}f(0, 0, 0) = (3, 0, 0) \cdot (2, 1, -2)/3 = 2.$$

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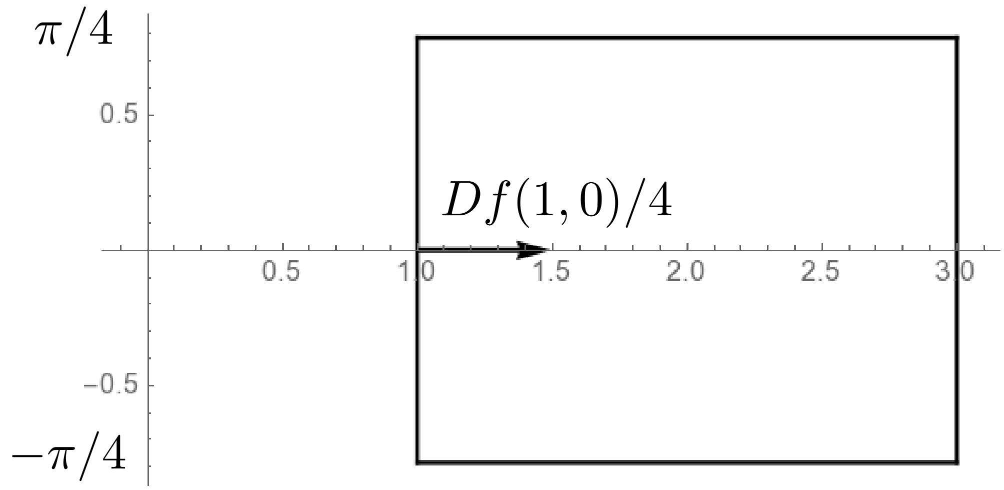
3. (14.7.37) Consider the function $f(x,y) = (4x - x^2)\cos y$ on the rectangle $\mathcal{U} = (1,3) \times (-\pi/4, \pi/4)$ and its closure $\bar{\mathcal{U}} = [1,3] \times [-\pi/4, \pi/4]$.
   (a) (10 points) Is the point $(1,0)$ a local minimum or a local maximum?

   (b) (10 points) Is the point $(2,0)$ a local minimum or a local maximum?

Solution:

(a) The point $(1,0)$ is on the boundary $\partial\mathcal{U}$ of $\mathcal{U}$, so we must examine the boundary behavior of f near $(1,0)$. On the other hand, the function is defined and differentiable in a larger domain, so we can get some information from the gradient as well. $Df = ((4-2x)\cos y, -(4x - x^2)\sin y)$, and $Df(1,0) = (2,0)$.



Notice that the gradient points into the rectangle. In particular, the values of the function increase at the rate 2 upon entering the rectangle at this direction. Thus, a maximum cannot occur at $(1,0)$.

On the other hand, along the boundary with $x = 1$, the function values are $f(1,y) = 3\cos y$ for $-\pi/4 < y < \pi/4$. This function has a maximum of 3 at $(1,0)$, and it follows that $(1,0)$ is not a local minimum of the function's values either.

Thus, the correct answer is "No, $(1,0)$ is not a local minimum nor a local maximum."

(b) $Df(2,0) = (0,0)$, so $(2,0)$ is an interior critical point. We calculate

$$D^2 f = \begin{pmatrix} -2\cos y & -(4-2x)\sin y \\ -(4-2x)\sin y & -(4x-x^2)\cos y \end{pmatrix} \quad \text{and} \quad D^2 f(2,0) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}.$$

Since $D^2 f(2,0)$ is a negative definite matrix, we know $(2,0)$ is a local maximum.

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4. (20 points) (14.8.29) A space probe in the shape of an ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

Solution: We use the method of Lagrange multipliers. Setting $g(x, y, z) = 4x^2 + y^2 + 4z^2$, we compute

$$DT + \lambda Dg = (16x, 4z, 4y - 16) + \lambda(8x, 2y, 8z).$$

Thus, we arrive at the system of four equations for (x, y, z) and λ:

$$\begin{cases} 2x + \lambda x = 0 \\ 2z + \lambda y = 0 \\ y + 2\lambda z = 4 \\ 4x^2 + y^2 + 4z^2 = 16. \end{cases}$$

From the first equation either $x = 0$ or $\lambda = -2$. If $x = 0$, the system reduces to three equations:

$$\begin{cases} 2z + \lambda y = 0 \\ y + 2\lambda z = 4 \\ y^2 + 4z^2 = 16. \end{cases}$$

Multiplying the first equation by $2z$ and the second equation by y, and then adding the results, we find $y^2 + 4z^2 + 4\lambda yz = 4y$. Replacing the first two terms according to the third equation gives

$$\lambda yz = y - 4 \qquad \text{or} \qquad \lambda z = \frac{y-4}{y}.$$

(Notice we can't have $y = 0$.) Substituting for λz in the second equation gives $y^2 + 2y - 8 = 4y$. That is, $y = -2$ or $y = 4$. If $y = -2$, then we get points $(x, y, z) = (0, -2, \pm\sqrt{3})$ for which

$$T(0, -2, \pm\sqrt{3}) = 600 \mp 24\sqrt{3}.$$

If $y = 4$, we get the point $(0, 4, 0)$, and $T(0, 4, 0) = 600$ (which is clearly neither a max or min value).

The other major possibility is $\lambda = -2$. In this case the remaining three equations are

$$\begin{cases} 2z - 2y = 0 \\ y - 4z = 4 \\ 4x^2 + y^2 + 4z^2 = 16. \end{cases}$$

Thus, $y = z = -4/3$ and $x^2 = 4 - 20/9 = 16/9$, so we consider also

$$T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 600 + \frac{128}{9} + 4\left(\frac{4}{3} + 4\right)\left(\frac{4}{3}\right) = 600 + \frac{128 + 256}{9}.$$

Finally, then we need to decide which is larger between

$$24\sqrt{3} \qquad \text{and} \qquad \frac{384}{9} = \frac{128}{3}.$$

Multiplying both numbers by $3/8$, we can compare $9\sqrt{3}$ and 16. Squaring we can compare 243 and 256. The second one is the maximum, so the maximum temperature is given by

$$T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 600 + \frac{384}{9}.$$

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5. (20 points) (14.4.59) If

$$f(x) = \int_0^x \sqrt{t^2 + x^2}\, dt,$$

find $f'(x)$.

Solution: Remember to look for the x dependencies. If we consider a function of two variables (a, b) given by

$$F(a, b) = \int_0^a \sqrt{t^2 + b^2}\, dt,$$

we see $f(x) = F(x, x)$, and we can use the chain rule:

$$\frac{df}{dx} = \frac{\partial F}{\partial a}\left(\frac{da}{dx}\right) + \frac{\partial F}{\partial b}\left(\frac{db}{dx}\right)$$

where $a(x) = x$ and $b(x) = x$. Thus,

$$f'(x) = \sqrt{2x^2} + \int_0^x \frac{x}{\sqrt{t^2 + x^2}}\, dt.$$

More details: I think the chain rule part of this problem is explained pretty clearly above. I don't really know how to explain it more clearly. If you are confused on that, please see what's above and think about it carefully.

What I'm thinking is that the real problem is with the differentiation. First of all, let's consider

$$\frac{\partial F}{\partial a} = \frac{\partial}{\partial a}\int_0^a \sqrt{t^2 + b^2}\, dt = \sqrt{a^2 + b^2}.$$

This is just one version of the fundamental theorem of calculus, which says if there is a variable in the upper limit of integration and you differentiate with respect to that variable, you just plug in the variable to the integrand:

$$\frac{d}{dx}\int_a^x g(t)\, dt = g(x).$$

If you don't get this, go back and review the fundamental theorem of calculus on pages 331 and 332 (chapter 5) of the Thomas text. Notice that when you put $a = b = x$, this result becomes $\sqrt{2x^2}$ which is the first term in the answer.

The second integral

$$\frac{\partial F}{\partial b} = \frac{\partial}{\partial b}\int_0^a \sqrt{t^2 + b^2}\, dt = \int_0^a \frac{b}{\sqrt{t^2 + b^2}}\, dt. \tag{1}$$

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is something called **differentiating under the integral sign**. This is explained on page 844 just before the referenced problem. It is not given a formal name there, but they simply say something like

$$\frac{d}{dx}\int_a^b g(x,t)\,dt = \int_a^b \frac{\partial g}{\partial x}(x,t)\,dt.$$

In short, you can just differentiate the integrand (under the integral sign) with the regular derivative changing to a partial derivative (under the integral sign). The formula (1) above follows because

$$\frac{\partial}{\partial b}\left(\sqrt{t^2+b^2}\right) = \frac{b}{\sqrt{t^2+b^2}}.$$