**Math 2550, Exam 2** practice                                        Name and section: ________________

1. (20 points) (14.2.40) Let

$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{for} \qquad (x, y) \neq (0, 0).$$

Show that the limit as $(x, y) \to (0, 0)$ of $f(x, y)$ does not exist.

**Solution:** It is enough to show limits from two different directions exist but have different values. If we take $x = 0$ but $y > 0$, then we find

$$\lim_{y \searrow 0} f(0, y) = 0.$$

If we take $y = 0$ by $x > 0$, then we find

$$\lim_{x \searrow 0} f(x, 0) = 1.$$

Since these two values are different, the limit does not exist.

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2. (a) (10 points) (14.3.6) Let $f(x,y) = (2x-3y)^3$. Find

$$\frac{\partial f}{\partial x} = \qquad\qquad\qquad \text{and} \qquad \frac{\partial f}{\partial y} =$$

(b) (10 points) (14.5.6) Let

$$g(x,y) = \tan^{-1}\left(\frac{\sqrt{x}}{y}\right).$$

Find $\nabla g$.

**Solution:**

(a)

$$\frac{\partial f}{\partial x} = 6(2x-3y)^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -9(2x-3y)^2.$$

(b)

$$\nabla g = \left(\frac{\frac{1}{2y\sqrt{x}}}{1+\frac{x}{y^2}}, \frac{-\frac{\sqrt{x}}{y^2}}{1+\frac{x}{y^2}}\right) = \left(\frac{y}{2\sqrt{x}(y^2+x)}, -\frac{\sqrt{x}}{y^2+x}\right).$$

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3. (20 points) (14.6.2) Find the the equation of the tangent plane to the surface

$$x^2 + y^2 - z^2 = 18$$

at the point $(3, 5, -4)$.

**Solution:** Letting $f(x, y, z) = x^2 + y^2 - z^2$, we have $Df = (2x, 2y, -2z)$. Therefore, $Df(3, 5, -4) = (6, 10, 8)$ is normal to the level surface and is a normal to the desired plane. The equation of the plane is

$$[(x, y, z) - (3, 5, -4)] \cdot (6, 10, 8) = 0 \qquad \text{or} \qquad 3x + 5y + 4z = 18.$$

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4. (20 points) (14.7.5) Find the local maxima of $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$ (considered with domain the entire plane $\mathbb{R}^2$).

**Solution:**

$$Df = (2y - 2x + 3, 2x - 4y).$$

Therefore, the gradient vanishes only at $(x,y) = (3, 3/2)$.

$$D^2 f = \begin{pmatrix} -2 & 2 \\ 2 & -4 \end{pmatrix}.$$

This is a negative definite matrix because the determinant is $8 - 4 = 4 > 0$. Thus,

$$f(3, 3/2) = 9 - 9 - 9/2 + 9 + 4 = 17/2$$

gives a local maximum.

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5. (20 points) (14.8.5) Find the points on the curve $xy^2 = 54$ closest to the origin.

**Solution:** We use the method of Lagrange multipliers. Setting $g(x, y) = xy^2$ we attempt to minimize

$$f(x, y) = \sqrt{x^2 + y^2}$$

subject to $g(x, y) = 54$. Accordingly, we compute

$$Df + \lambda Dg = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) + \lambda(y^2, 2xy).$$

Notice that $y \neq 0$ on the constraint curve, so $Dg \neq (0, 0)$ there. The three equations for $(x, y)$ and $\lambda$ are

$$\begin{cases} x + \lambda y^2 \sqrt{x^2 + y^2} = 0 \\ y + 2\lambda xy \sqrt{x^2 + y^2} = 0 \\ xy^2 = 54. \end{cases}$$

Multiplying the first equation by $x$, multiplying the second equation by $y/2$ and then subtracging the results, we get

$$x^2 - \frac{y^2}{2} = 0.$$

This means $y^2 = 2x^2$ and (from the third equation) $2x^3 = 54$. Thus, the extreme points all have $x = 3$. The third equation then gives $y = \pm 3\sqrt{2}$. The points $(3, \pm 3\sqrt{2})$ are equidistant from the origin, and the minimum distance possible must be $3\sqrt{3}$ attained at these points.

There are a couple things one might nominally worry about here. The first is that rather than closest points, we might have found farthest points. The other is that the curve somehow spirals and there are no closest points. There are various ways to see that nothing like this happens, and we've got the correct solution. Probably the easiest is to simply note that the level curve in question is given by $x = 54/y^2$ which is a graph over the $y$-axis and is easy to sketch. A sketch of a very similar level curve was given in the other practice exam coming from problem 14.5.3: