The Math 2550, Introduction to Multivariable Calculus
Recitation Worksheets

Contents

About This Document

This document is a collection of exercises that are meant for students to work on during Math 2550 recitations, held once per week. These worksheets were developed for a 12 week course, with 3 midterms, each recitation was 70 minutes long. Each worksheet provides problems for one recitation, although there may not be time for students to complete all problems.

I Worksheet 1

Sections from Thomas 13^{th} edition: 12.4, 12.5, 12.6, 13.1

Exercises

1. A plane is a set of points that satisfies an equation of the form $c_1x + c_2y + c_3z = c_4$.
   (a) Find any three distinct points, P, Q, R, in the plane $2x + 3y + z = 0$.
   (b) Use a cross product to construct a vector perpendicular to the plane.
   (c) How is the vector you created related to the vector made from the coefficients of x, y, and z in the equation for the plane, which is $2\hat{\imath} + 3\hat{\jmath} + \hat{k}$?
   (d) Find any point that is not in the plane, S, and then find the closest point on the plane to S.
2. Calculate the cosine of the angle between the vector $\vec{x} = \hat{\imath} + 2\hat{k}$ and the plane $2x + y - z = 0$. How are the plane and the vector related to each other, geometrically? Why?
3. Find the distance between the point (1,2,3) and the line $\vec{r} = \hat{\imath} + 2\hat{k} + t(\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$.
4. Consider the line $\langle -1, t, 1\rangle$ and the plane $2x + y - z = 3$, where $t \in \mathbb{R}$. Find all values of t so that the distance between points on the line and the plane is $2\sqrt{6}$.
5. Consider the points $P(1, 0, 3), Q(2, 2, 3), R(0, 0, -1)$.
   (a) Find a vector that is perpendicular to the plane that passes through these points.
   (b) Find a point S so that a unique plane that passes through P, Q, and S cannot be found. Describe why you cannot find a unique plane that passes through P, Q, and your point, S.
6. Find the equation of the plane that passes through the point (-1,2,1) and contains the line $x = y = z$.
7. Suppose we have two planes $2x+y-z = 3$ and $x+3y+z = 0$. Find the line of intersection between these two planes, and find the equation of the plane that passes through the line of intersection and through the point $(0, 0, 0)$.

Group Work Problems

1. Consider the set of all points that are equidistant from the point $(0, 1, 0)$ and the plane $z = 1$.
   (a) Find an equation that represents the surface.
   (b) Briefly describe the surface in words.
   (c) Give a rough sketch of the surface.
2. Consider the surfaces $z = \sqrt{4 - x^2 - y^2}$ and $y^2 + x^2 - 2y = 0$.
   (a) Find a parametric vector representation, $\vec{r}(t)$, of the curve that satisfies both equations.
   (b) Give a rough sketch of the surfaces and the curve in one plot.

Additional Problems (if time permits)

1. Find the area of the triangle with the vertices (6,3), (4,5), and (3,4).
2. Indicate which of the following are always true.
   (a) $\vec{u} \cdot \vec{u} = |\vec{u}|$
   (b) $\vec{u} \times (-\vec{u}) = \vec{0}$
   (c) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$
3. Which of the following make sense? Explain why/why not.
   (a) $\vec{a} \times (\vec{b} \cdot \vec{c})$
   (b) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$
   (c) $\vec{a} \times (\vec{b} \times \vec{c})$
   (d) $\vec{a} \cdot (\vec{b} \times \vec{c})$
4. Consider the surface $z = ax^2 + by^2$, where a and b are constants. Identify all possible surfaces for the following cases, by stating the name of the surface and describing its orientation, if applicable.
   (a) $ab > 0$
   (b) $ab < 0$
   (c) $a = b = 0$
5. The path of an object is given by $\vec{r} = 2t\hat{i} + t^2\hat{j}$ for $t \geq 0$. Sketch the curve in the xy-plane and indicate the direction of motion.
6. S is the surface described by the equation $x^2 - 2x - y - z^2 - 4z = 3$. Create a rough sketch of S and name the surface.

Partial Solutions to WS 1

1. (a) If we choose $x = 0, y = 0$, then $z = 0$, so $P(0, 0, 0)$.
   If we choose $x = 1, y = 0$, then $z = -2$, so $Q(1, 0, -2)$.
   If we choose $x = 0, y = 2$, then $z = -3$, so $R(0, 1, -3)$.

   (b) A vector perpendicular to plane is $\vec{N} = \vec{PQ} \times \vec{PR} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$. Any scalar multiple of this vector is acceptable.

   (c) In this case they are equal to each other. More accurately, they are scalar multiples of each other.

   (d) A point not on the plane is $S(0, 0, 1)$. The distance between S and the plane is $\left| \vec{PS} \cdot \frac{\vec{N}}{|\vec{N}|} \right|$

2. Let the vector perpendicular to the plane be $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. The angle, θ, between the given vector and $\mathbf{y}$ is found by solving

$$\begin{aligned} \cos\theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||} \\ &= \frac{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{5}\sqrt{6}} \\ &= \frac{0}{\sqrt{5}\sqrt{6}} \\ &= 0. \end{aligned}$$

   The angle, θ, is $\pi/2$. But θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ and we need the angle between $\mathbf{x}$ and the plane, which is $\pi/2 - \theta = 0$. Therefore, the desired angle is 0 (the given vector is parallel to the plane).

3. **Distance Between a Point and a Line**
   A formula we can use is

$$D = \frac{||\vec{v} \times \vec{w}||}{||\vec{v}||}$$

   where $\vec{v}$ is a vector parallel to the line, $\vec{v} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. The vector $\vec{w}$ is the directed line segment from a point on the line to the given point

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

   Using these vectors in the distance formula yields $D = \sqrt{69/14}$.

4. **Distance Between a Point and a Plane**
   A formula we can use is

$$D = \left| \frac{\overrightarrow{PS} \cdot \vec{n}}{|\vec{n}|} \right|$$

   where the normal vector to the plane is $\vec{n} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$. The point $P(0, 3, 0)$ is in the plane, so $\overrightarrow{PS} =$

$-\hat{\imath} + (t-3)\hat{\jmath} + \hat{k}$, and $D = 2\sqrt{6}$. Thus,

$$\begin{aligned} D = 2\sqrt{6} &= \frac{|2(-1) + (t-3) + (1)(-1)|}{\sqrt{6}} \\ &= \frac{|t-6|}{\sqrt{6}} \\ 12 &= |t-6| \end{aligned}$$

Therefore, $t = 18$ or $t = -6$.

5. (a) We can start by finding the plane that contains the three points. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \overrightarrow{PR} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$$

A vector perpendicular to the plane that contains the three points is found by calculating the cross product between these two vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (-8-(0))\mathbf{i} - (-4-0)\mathbf{j} + (0-(-2))\mathbf{k} = -8\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

Any vector parallel to this vector is perpendicular to the plane that contains the given points.

(b) If the point S is such that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{QS}$, then there will be an infinite number of planes that pass through these three points. Such a point, S, can be determined by multiplying $\overrightarrow{PQ}$ by a constant. Choosing 2 as a constant, then

$$\overrightarrow{QS} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

We determine the components of S as

$$S = \begin{bmatrix} 2-2 \\ 4-2 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

6. The points $(0,0,0)$ and $(1,1,1)$ are on the given line, so the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is parallel to the desired plane.

Another vector parallel to the plane is

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, a vector perpendicular to the desired plane is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = (1-2)\mathbf{i} - (1+1)\mathbf{j} + (2+1)\mathbf{k} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}.$$

The equation of the desired plane is

$$0 = -1(x+1) - 2(y-2) + 3(z-1) = -x - 2y + 3z.$$

7. Let the normal vector of the first plane be $\mathbf{n}_1$, and the normal vector of the second plane be $\mathbf{n}_2$. Then

$$\mathbf{n}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The line that intersects these two planes is a line that is in both of the planes, and therefore must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore, the line we need is parallel to the vector, $\mathbf{a}$, given by the cross product

$$\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (4, -3, 5).$$

This vector is parallel to the desired plane. To find a second vector in the desired plane, we can find a line from the given point to any point in the line of intersection of the two given planes. Letting $y = 0$, we obtain the equations

$$\begin{aligned} 2x - z &= 3 \\ x + z &= 0 \end{aligned}$$

which has the solution $x = 1, z = -1$. Therefore, the point $(1, 0, -1)$ is in the intersection of the two planes, and a vector in the plane is

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

A normal vector to the desired plane is

$$\begin{aligned} [1, 0, -1]^T \times [4, -3, 5]^T &= (0 - 3)\mathbf{i} - (5 + 4)\mathbf{j} + (-3 - 0)\mathbf{k} \\ &= -3\mathbf{i} - 9\mathbf{j} - 3\mathbf{k} \end{aligned}$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= (-3)(x - 0) + (-9)(y - 0) + (-3)(z - 0) \\ &= -3x - 9y - 3z \end{aligned}$$

Group Work Problems

1. The distance, d_1, from the point (x, y, z) and the point $(0, 1, 0)$ is given by

$$d_1 = \sqrt{x^2 + (y - 1)^2 + z^2}.$$

The distance, d_2, from the point (x, y, z) and the given plane is

$$d_2 = \big|z - 1\big|.$$

The points that are equidistant from the point $(0, 1, 0)$ and the given plane are the points that such that $d_1 = d_2$. Therefore,

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y - 1)^2 + z^2} &= \big|z - 1\big| \\ x^2 + (y - 1)^2 + z^2 &= (z - 1)^2 \\ x^2 + (y - 1)^2 + z^2 &= z^2 - 2z + 1 \\ x^2 + (y - 1)^2 &= -2z + 1 \\ z &= -\frac{1}{2}\Big(x^2 + (y - 1)^2 - 1\Big) \end{aligned}$$

The surface is an elliptic paraboloid that opens downward and whose vertex is located at the point $(0, 1, 1/2)$.

2. Below is a screen capture of hand written solutions.

$$\underbrace{z=\sqrt{4-x^2-y^2}}_{1},\ \underbrace{y^2+x^2-2y=0}_{2}$$

We want $x(t)$, $y(t)$, $z(t)$ that satisfy given equations, 1 and 2

Complete the square:

$x^2+y^2-2y+1-1=0$

$x^2+(y-1)^2=1$

$\Rightarrow x=\cos t,\ y=\sin t+1$ then 2 is satisfied.

$z=\sqrt{4-c^2-(1+s)^2}=\sqrt{2-2\sin t} \Rightarrow \vec{r}=\begin{bmatrix}\cos t\\ \sin t+1\\ \sqrt{2-2\sin t}\end{bmatrix}$

CURVE IS INTERSECTION OF $\underbrace{x^2+y^2+z^2=4}_{\text{SPHERE}}$ AND $\underbrace{x^2+(y-1)^2=1}_{\text{CYLINDER}}$

Additional Problems

1. Let the points be $P(6,3), Q(4,5), R(3,4)$. Then $\overrightarrow{PQ} = \begin{bmatrix}-2\\2\\0\end{bmatrix}$, $\overrightarrow{PR} = \begin{bmatrix}-3\\1\\0\end{bmatrix}$, $\overrightarrow{PQ} \times \overrightarrow{PR} = 4$, so the triangle has area 2.

2. (a) $\vec{u}\cdot\vec{u} = |\vec{u}|$ is never true. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
   (b) $\vec{u}\times(-\vec{u}) = \vec{0}$ is always true because the two vectors are co-linear.
   (c) $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$ is always true because the $(\vec{u}\times\vec{v})$ is perpendicular to $\vec{u}$.

3. (a) $\vec{a}\times(\vec{b}\cdot\vec{c})$ is not defined
   (b) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is not defined
   (c) $\vec{a}\times(\vec{b}\times\vec{c})$ is defined
   (d) $\vec{a}\cdot(\vec{b}\times\vec{c})$ is defined

4. (a) If $ab > 0$, then a and b have the same sign. If they are both positive, the surface is an elliptic paraboloid that opens upward. If a and b are both negative, the surface is an elliptic paraboloid that opens downward.
   (b) If $ab < 0$, then a and b have opposite signs (i.e. - one is positive, the other is negative). The surface is a hyperbolic paraboloid.
   (c) If $a = b = 0$, then $z = 0$.

5. $x(t) = 2t$, $y = t^2 = (x/2)^2$, for $t \geq 0$. The plot should be one half of a parabola, with the object moving away from the origin.

6. Completing the square, we can express S as $(x-1)^2 - (z+2)^2 - y = 0$. The surface is a hyperbolic paraboloid.

II Worksheet 2

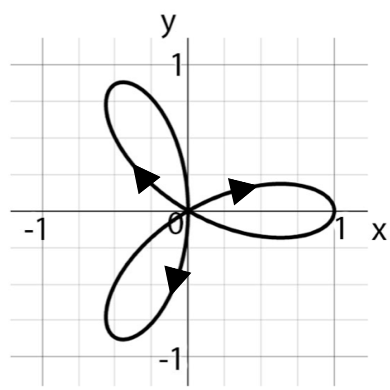
Sections from Thomas 13^{th} edition: 13.2, 13.3, 13.4

Exercises

1. An object is moving along the curve $\vec{r}(t) = 5\sin(\pi t)\hat{\imath} + 5\cos(\pi t)\hat{\jmath} + 12\pi t\hat{k}$, $t \in \mathbb{R}$. Where could the object be, if it has travelled a distance of 39π units along the curve, starting from $P(0, -5, -12\pi)$?

2. An object is moving in the yz-plane. At point P, its tangent vector is $\vec{T} = \hat{k}$. What could the normal vector be equal to at P? List all possibilities. *Note: a similar question was on a Spring 2016 final exam.*

3. An object is moving in the xy-plane along the curve shown in the graph. The direction of motion is indicated with arrows. Identify the points where the curvature has a local maximum. Sketch the normal and tangent vectors at those points.

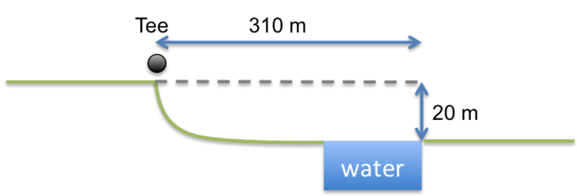


4. An object is moving along the curve $\vec{r}(t) = -9\ln(\sec t)\hat{\imath} - 9\hat{\jmath}$, for $0 \leq t \leq \pi/4$.
   (a) Compute the arc length parametrization.
   (b) Compute the unit tangent vector.
   (c) Compute the curvature.

5. A ball is thrown from a height of 3 meters above the ground, at a speed of $2\sqrt{2}$ m/s, at an angle of $45°$ above the horizontal. A constant wind is blowing that adds a component of $4\hat{\imath}$ m/s^2 to the ball's acceleration while it is in the air. Where does the ball land? You may use $g = 10$ m/s^2.

Group Work Problems

1. A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water? *You may use g = 10 m/s^2, but either way you may want to use a calculator.*



2. An object is moving along the intersection of the plane $y = 2$ and the surface $z = x^2 + y^2$.
   (a) Find a parametric representation for the motion.

(b) Sketch of the plane, surface, and the curve. Without any calculation, indicate the point(s) where you think the curvature, κ, of your curve is maximized, and sketch the normal and tangent vectors at that point.

(c) Verify your answer to the previous question by finding the point of maximum κ, and calculate the tangent vector at that point. Hint: some curvature formulas require fewer calculations than others.

(d) Find the equation of the osculating circle at the point where κ is maximum.

Additional Problems (if time permits)

1. Without any calculation, what is the value of the normal vector in the second group work problem? Calculate the normal vector for this problem when $t = 0$ to verify your hypothesis.

Partial Solutions to Worksheet 2

1. First note that at the point P, $t = -1$. For convenience, let $S = \sin(\pi t)$ and $C = \cos(\pi t)$.

$$\begin{aligned}
\vec{r}(t) &= 5S\hat{\imath} + 5C\hat{\jmath} + 12\pi t\hat{k} \\
\vec{v}(t) &= 5\pi C\hat{\imath} - 5\pi S\hat{\jmath} + 12\pi\hat{k} \\
|\vec{v}(t)| &= \pi\sqrt{5^2\left(S^2 + C^2\right) + 12^2} \\
&= \pi\sqrt{5^2 + 12^2} \\
&= 13\pi
\end{aligned}$$

We want the points that are 39π units away from P along the curve. The distance between P and a point on the curve $\vec{r}$ at t is

$$s(t) = \int_{-1}^{t} 13\pi \, d\tau = 13\pi(t+1)$$

We need the values of t that yield a distance of 39π units away, which we find by solving

$$\begin{aligned}
39\pi &= |13\pi(t+1)| \\
3 &= |t+1| \\
t &= -4, 2.
\end{aligned}$$

There are two points 39π units away, $(0, 5, -48\pi)$, and $(0, 5, 24\pi)$.

2. When the object is turning,

   - $\vec{N}(t)$ indicates the direction that the object is turning,
   - $\vec{N}(t)$ is perpendicular to $\vec{T}$, and
   - $\vec{N}(t)$ has unit length.

   Let $\vec{N}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. The object is moving in the yz-plane, so $x(t) = 0$ for all t. If $\vec{N}$ is perpendicular to $\vec{T}$ at P, $0 = \vec{N} \cdot \vec{T} = z$, so $z = 0$ at P. So $\vec{N} = y(t)\hat{\jmath}$. But $\vec{N}$ also has unit length, so $y = \pm 1$. So if the object is turning at P, $\vec{N} = \pm\hat{\jmath}$.

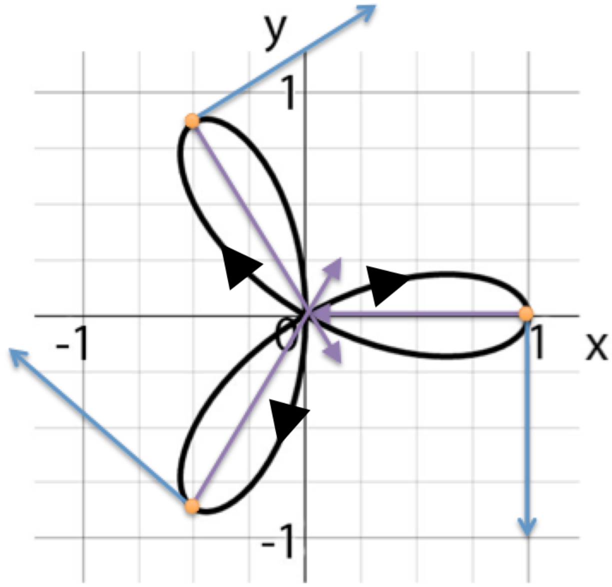
   But wait! What happens if the object is not turning? The normal vector is undefined in this case.

   The normal vector can be either: undefined, $\hat{\jmath}$, or $-\hat{\jmath}$

3. Vectors $\vec{T}$ and $\vec{N}$ should be drawn in a way that they have the following properties.

   (a) $\vec{T}$ and $\vec{N}$ should be perpendicular to each other
   (b) $\vec{T}$ and $\vec{N}$ should have unit length
   (c) $\vec{T}$ should point in the direction of motion
   (d) $\vec{N}$ should point in the direction that the object is turning

   The curvature is has three local maxima, indicated below with orange dots, along with a rough sketch of vectors $\vec{T}$ and $\vec{N}$ at those three points. The blue vectors are the tangent vectors, the purple vectors are the normal vectors.



4. First compute the velocity and speed.

$$\begin{aligned} \vec{v} &= -9\frac{1}{\sec t}\frac{d}{dt}\sec t\,\hat{\imath} - 0\,\hat{\jmath} = -9\tan t\,\hat{\imath} \\ \text{speed} &= 9\tan t \end{aligned}$$

   (a) Integrating the speed from 0 to t gives us the arc length parametrization.

$$L(t) = \int_0^t \text{speed}\,d\tau = \int_0^t 9\tan\tau\,d\tau = 9\ln(\cos(\tau))\Big|_0^t = 9\ln(\cos t)$$

   (b) The unit tangent vector, $\vec{T}$ is

$$\vec{T} = \vec{v}/|\vec{v}| = -\hat{\imath}$$

   (c) The curvature is zero, because the tangent vector is constant for all values of t.

5. Constructing the acceleration vector $\vec{a} = 4\hat{\imath} - 10\hat{\jmath}$, we integrate twice and use the initial conditions to find the position when the ball lands.

$$\begin{aligned} \vec{a} &= 4\hat{\imath} - 10\hat{\jmath} \\ \vec{v} &= (4t + c_1)\hat{\imath} + (-10t + c_2)\hat{\jmath} \\ \vec{v}(0) &= 2\sqrt{2}(\cos(\pi/4)\hat{\imath} + \sin(\pi/4)\hat{\jmath}) = 2(\hat{\imath} + \hat{\jmath}) \\ \vec{v} &= (4t + 2)\hat{\imath} + (-10t + 2)\hat{\jmath} \\ \vec{r} &= (2t^2 + 2t + d_1)\hat{\imath} + (-5t^2 + 2t + d_2)\hat{\jmath} \\ \vec{r}(0) &= 3\hat{\jmath} \Rightarrow d_1 = 0, d_2 = 3 \\ \vec{r} &= (2t^2 + 2t)\hat{\imath} + (-5t^2 + 2t + 3)\hat{\jmath} \end{aligned}$$

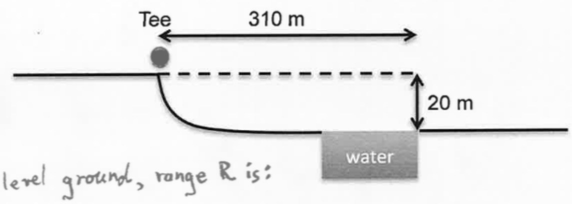
The ball lands when the vertical component is zero, or when $t = 1$. Using this time in the position vector above, we find that the ball lands at the point (4, 0).

Partial Solutions to Group Work Problems

1. Below is a screen capture of hand written solutions.

$\vec{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$

$\vec{v} = \begin{bmatrix} c_1 \\ -gt + c_2 \end{bmatrix} = \begin{bmatrix} v_0 \cos\alpha \\ -gt + v_0 \sin\alpha \end{bmatrix}$



Use range formula to get v_0. On level ground, range R is:

$$R = 300 = \frac{v_0^2 \sin 2\alpha}{g}$$

Range R maximized when $\alpha = \pi/4$, solving for v_0 yields $v_0 = \sqrt{300g} \approx 54.25$.

Velocity $\vec{v}$ becomes: $\vec{v}(t) = \begin{bmatrix} \sqrt{300g}\,(1/\sqrt{2}) \\ -gt + \sqrt{300g}\,\frac{1}{\sqrt{2}} \end{bmatrix}$.

Position $\vec{r}(t) = \begin{bmatrix} v_0 \cos\alpha\, t + d_1 \\ -g\frac{t^2}{2} + v_0 \sin\alpha\, t + d_2 \end{bmatrix}$, but we'll use $d_1 = d_2 = 0$.

x-component is 310 when: $310 = v_0 \cos\alpha\, t$, solving for t yields $t \approx 8.08\ldots$

We need y-component to be > -20 when $t \approx 8$; $-g(5.7)^2/2 + \sqrt{300g}\,(\sin\frac{\pi}{4})\,5.7$

≈ -11

$\Rightarrow$ YAY! Golfer can clear water.

2. (a) Let $x = t$, $y = 2$, and $z = t^2 + 4$.

$$\vec{r} = t\hat{\imath} + 2\hat{\jmath} + (t^2 + 4)\hat{k}$$

Any parametrization that satisfies the given equations is valid.

(b) The point of maximum curvature is at the vertex of the parabola, which is when $t = 0$. The unit tangent vector will be tangent to the curve and pointing in the direction of motion, which at the vertex is only in the $\hat{\imath}$ direction, and $\vec{T}$ has unit length, so $\vec{T}(0) = \hat{\imath}$. The normal curve will be perpendicular to $\vec{T}$, also has unit length, points in the direction that it is turning, so $\vec{N}(0) = \hat{k}$.

(c) Note that the curve is a parabola in the $y = 2$ plane, with equation $z = t^2 + 4 = x^2 + 4$.

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}} = \frac{|z''(x)|}{\left(1 + (z'(x))^2\right)^{\frac{3}{2}}} = \frac{2}{\left(1 + (2x)^2\right)^{\frac{3}{2}}}$$

By inspection, the curvature has a maximum when $x = 0$. The tangent vector is

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{\imath} + 2t\hat{\jmath}}{\sqrt{1 + 4t^2}}$$

So at $t = 0$, $\vec{T}(0) = \hat{\imath}$, as we expected.

(d) The circle passes through the point $P(0, 2, 4)$, lies above the parabola, and has radius $1/\kappa(0) = 1/2$, so the centre of the circle is at the point $Q(0, 2, 4.5)$. The circle has equation $x^2 + (z - 4.5)^2 = \frac{1}{2^2}$.

Partial Solutions to Additional Problems

1. The normal vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

We first obtain the derivative of $\vec{T}$.

$$\frac{d}{dt}\vec{T}(t) = \frac{d}{dt}\frac{\hat{i} + 2t\hat{k}}{\sqrt{1+4t^2}} = \frac{-4t}{(1+4t^2)^{3/2}}\hat{i} + \frac{2}{(1+4t^2)^{3/2}}\hat{k} = (1+4t^2)^{-3/2}(-4t\hat{i} + 2\hat{k})$$

We are interested in what happens when $t = 0$, and $\vec{T}'(0) = 2\hat{k}$.

$$\left|\frac{d}{dt}\vec{T}(t)\right| = \left|(1+4t^2)^{-3/2}(-4t\hat{i} + 2\hat{k})\right| = \left|(1+4t^2)^{-3/2}\right|\sqrt{16t^2+4}$$

This quantity is equal to 2 when $t = 0$. Thus,

$$\vec{N}(0) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{2\hat{k}}{2} = \hat{k}$$

We obtained $\vec{N}(0) = \hat{k}$, as expected.

III Worksheet 3

Sections from Thomas 13^{th} edition: 14.1, 14.2, 14.3

Exercises

1. For the following function, a) identify and sketch the domain, b) state the boundary of the domain, c) indicate whether the domain is open and/or closed, and d) determine the range of $g(x, y)$.

$$g(x,y) = \frac{\sqrt{y+1}}{x^2y + xy^2}$$

2. (a) Give an example of a function whose domain is neither open nor closed.
   (b) Give an example of a function of two variables, $f(x, y)$, whose level curves, $C = f(x, y)$, are a family of parabolas that are symmetric about the y-axis.
   (c) Give an example of a function of two variables, $g(x, y)$, whose domain is an open set, and whose level curves, $C = g(x, y)$, are straight lines with slope C.
   (d) Give an example of a function of three variables, $h(x, y, z)$, whose level surfaces are a family of cones whose vertices are located at the origin.

3. An object is moving along the intersection of the plane $y = 3$ and the surface $f(x, y) = x^2 + y^2$. Sketch the plane, surface, and path. Find the equation of the tangent line to the path at the point $P(1, 3, 10)$, express the line in parametric vector form, and add the tangent line to your sketch.

4. Question 66 from Section 14.3, which is: find the value of $\frac{\partial x}{\partial z}$ at $(1, -1, -3)$ if $xz + y \ln x - x^2 + 4 = 0$ defines x as a function two independent variables y and z.

5. Show that the limit does not exist.

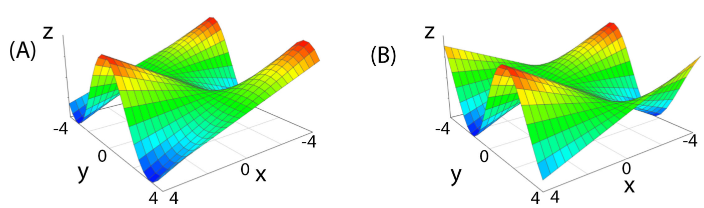
$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Group Work Problems

1. Identify where $f(x, y)$ is continuous, where

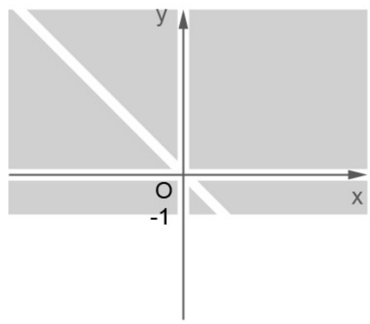
$$f(x,y) = \begin{cases} \frac{x^2-(y-1)^2}{x^2+(y-1)^2} & \text{if } (x,y) \neq (0,1) \\ 0 & \text{if } (x,y) = (0,1) \end{cases}$$

2. Shown below is a function, $f(x, y)$, and $f_y(x, y)$, on the domain $-4 \leq x \leq 4$, $-4 \leq y \leq 4$.
   i) Which of the two surfaces is $f_y(x, y)$?
   ii) Explain your reasoning.



Partial Solutions to Worksheet 3

1. (a) For $g(x,y)$ to be defined, its denominator cannot be zero. This implies that $0 \neq x^2y+xy^2 = xy(x+y)$. Thus, $x \neq 0$, $y \neq 0$, and $y \neq -x$. The numerator of $g(x,y)$ also cannot be complex, which implies that $y+1 \geq 0$, or that $y \geq -1$. The domain is the set $D = \{(x,y)|y \geq -1, x \neq 0, y \neq 0, y \neq -x\}$.



   (b) A boundary of a region is the set of boundary points, and a point is a boundary point if every ball centered on that point contains points inside the region and points outside the region. So the lines $y = -1, x = 0, y = 0, y = -x$ are all boundaries of the domain, and the boundary of the domain is the set of these lines.

   (c) A region is closed if it contains its entire boundary, which the domain does not. So the domain is not closed.

   A region is open if it only consists of its interior points. But the domain contains the boundary line $y = -1$, so the region is also not open.

   The entire domain, D, is neither closed nor open.

   (d) The numerator is always non-negative, and the denominator can be any real number. The range is the set of all real numbers (if you aren't yet convinced, try setting $y = constant$ and see what you get).

2. (a) One example is $f(x,y) = \frac{\sqrt{x-1}}{y+1}$.

   (b) Parabolas whose vertices are at the point $(0, C)$ can be expressed as

$$y = C - x^2$$

   Solving for C, we obtain

$$C = y + x^2$$

   A function with the desired properties is

$$f(x,y) = y + x^2$$

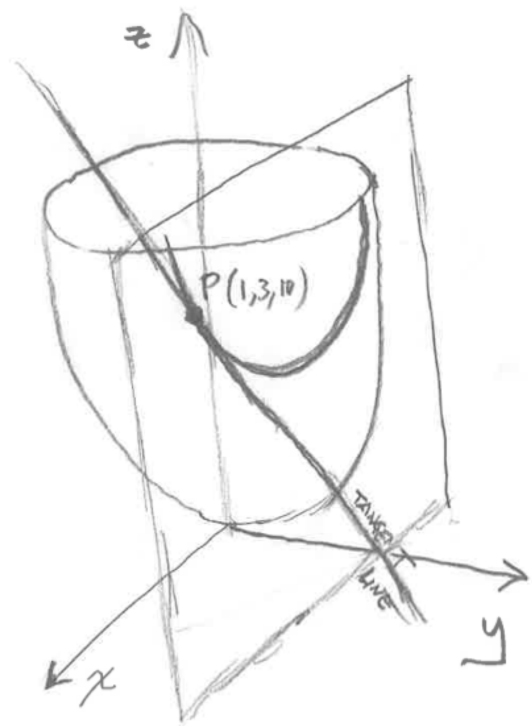
   There are many other acceptable solutions to this problem.

   (c) Straight lines with slope C have the form $y = Cx$. A function that meets the given criteria is $f = y/x$. The domain is the open set $x \neq 0$.

   (d) Cones with vertex at the origin have the form $z^2 = C(x^2 + y^2)$. A function that meets the given criteria is

$$h = \frac{z^2}{x^2 + y^2}$$

3. $f_x(x, y) = 2x$, $f_x(1, 3) = 2$, so equations for tangent line are $y = 3, z - 10 = 2(x - 1)$, or $z = 2x + 8$. In parametric form, we can let $x = t$, and represent the tangent line as $\vec{r}(t) = t\hat{\imath} + 3\hat{\jmath} + (8 + 2t)\hat{k}$, for parameter t. Alternatively, if we want to use the point $P(1, 3, 10)$ in our definition of the line, we could also use $\vec{l}(s) = (1 + s)\hat{\imath} + 3\hat{\jmath} + (10 + 2s)\hat{k}$, for parameter s. These representations of the tangent line are equivalent.



Note: be careful to not confuse the tangent line with the tangent vector. The tangent vector, $\vec{T}$, to our path, $\vec{r}(t) = t\hat{\imath} + 3\hat{\jmath} + (t^2 + 9)\hat{k}$, at any point, is

$$\vec{T}(t) = \frac{velocity}{speed} = \frac{\vec{v}}{|\vec{v}|} = \frac{r'(t)}{|r'(t)|} = \frac{\hat{\imath} + 2t\hat{k}}{\sqrt{1 + 2^2t^2}}$$

At P, $t = 1$, and the tangent vector, $\vec{T}(1)$, is

$$\vec{T}(1) = \frac{\hat{\imath} + 2\hat{k}}{\sqrt{3}}$$

4. Implicit differentiation:

$$\frac{\partial x}{\partial z}z + x + \frac{y}{x}\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$$
$$(z + y/x - 2x)\frac{\partial x}{\partial z} = -x$$

At the given point, $\frac{\partial x}{\partial z} = 1/6$.

5. (a) When we evaluate $f(x, y)$ at the limit point, we find $f(1, 0)$ is an indeterminant form of type $0/0$. It may be that f is not continuous at the point $(1, 0)$. In one dimension, we would use l'Hospital's rule

to evaluate such a limit. But l'Hospitals rule only works for functions of one variable, and we have a multivariable function. So for this problem, we will try approaching the limit point along curves that pass through the limit point. We can try evaluating the limit along $y = m(x-1)$, which passes through $(1,0)$.

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3} = \lim_{x\to 1} \frac{x(x-1)^3 + m^2(x-1)^2}{4(x-1)^2 + 9m^3(x-1)^3} = \lim_{x\to 1} \frac{x(x-1) + m^2}{4 + 9m^3(x-1)} = \frac{m^2}{4}$$

Because the value of the limit depends on the path of approach, the limit does not exist.

(b) Along the path $y = mx$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{xm^2x^2}{x^2+m^4x^4} = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2} = 0.$$

We might be tempted to believe that this limit exists. But along the path $x = my^2$, we find

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{my^4}{m^2y^4+y^4} = \frac{m}{m^2+1}$$

Because the value of the limit depends on the path of approach, the limit does not exist.

Partial Solutions to Group Work Problems

1. Our given function, f, is a rational function, everywhere, except at (0,1). Rational functions are continuous everywhere on their domain, and

$$\frac{x^2-(y-1)^2}{x^2+(y-1)^2}$$

has the domain $D = \{(x,y) \,|\, (x,y) \neq (0,1)\}$. For a function of two variables, $f(x,y)$, to be continuous at a point (x_0, y_0), we need

$$f(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

This is the definition of continuity at a point. There are a few ways that we can show that this particular limit does not exist. One strategy is to compare the limits along the lines $x = 0$ and $y = 1$. Along $x = 0$, we have

$$\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{y\to 1} \frac{0-(y-1)^2}{0+(y-1)^2} = -1$$

It does not matter what happens at the limit point, the function approaches -1 as y tends to 1. Along $y = 1$ we have

$$\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{x\to 0} \frac{x^2-0}{x^2+0} = 1$$

We obtained two different values for the limit when approaching along two different paths, so the limit does not exist. Therefore, the function is not continuous at (0,1), and is continuous elsewhere.

Another more general strategy is to consider paths of the form $y = kx + 1$, or $y - 1 = kx$. We would obtain

$$\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{x\to 0} \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$$

Because the result depends on k, the limit does not exist.

Regardless of which strategy you decide to use, be sure to consider paths that approach the limit point.

2. There are a few different ways to approach this problem. One strategy is consider the behaviour of the functions at a point, and another strategy is to consider the behaviour of the functions along lines.

    - **At a point**: consider the point where $x = 4, y = -4$. If (A) is $f(x, y)$, then the rate of change of $f(4, -4)$ in the y-direction is negative. But at $x = 4, y = -4$, the surface in (B) is not negative. If (B) is $f(x, y)$, then the rate of change in the y-direction is negative at $x = 4, y = -4$, and the surface at that point is also negative. So (B) is f, and (A) is f_y.
    - **Along a line**: consider the line $x = 4$. If (B) is $f(x, y)$, then the slope of the surface in the y direction is negative when $y = -4$, then as y increases it becomes positive, then negative. This behaviour is consistent in (A) along the line $x = 4$. However if (A) were $f(x, y)$, then the slope of the surface in the y direction starts negative, then becomes positive, then negative. But the value of the surface in (B) is positive at $(4, -4)$. So (B) must be $f(x, y)$.

    You might be wondering about how to deal with not having values labeled on the z-axis. Keep in mind that f_y must have positive and negative values, because the rate of change in the y-direction of $f(x, y)$ has positive and negative values.

IV Worksheet 4

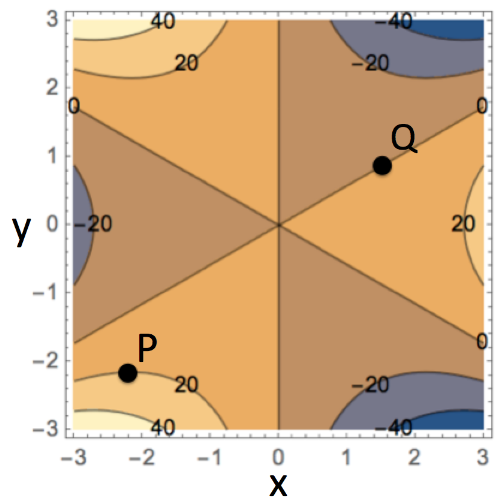
Sections from Thomas 13^{th} edition: 14.4, 14.5, 14.6

Exercises

1. Below is a contour map for a function $f(x, y)$. At points P and Q, draw the normalized gradient vector

$$\vec{g}(x, y) = \frac{\nabla f}{|\nabla f|}$$

Note: this problem is based on a question from a 2015 Math 2401 quiz created by Dr. Tom Morley.



2. Consider $f(x, y, z) = xz + y^2$.
   (a) Determine the maximum rate of change of $f(x, y, z)$, and the direction in which it occurs, at the point $(1, 1, 2)$.
   (b) If $x = \cos t$, $y = t + 1$, $z = t + 2$, use the chain rule to find $\frac{d}{dt} f(x(t), y(t), z(t))$ at the point $(1, 1, 2)$.
   (c) Approximate the value of $f(1, 0.9, 2.1)$ with a linear approximation.

3. Find the directional derivative of $f = z \ln(x/y)$ at $P(1, 1, 2)$ towards the point $Q(2, 2, 1)$. Provide a geometric interpretation of your derivative.

4. The radius of a cylinder is decreasing at a rate of 2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume changing when the radius is 10 cm and the height is 100 cm?

5. Calculate du/dt given that $u = x^2 - y^2$, $x = t^2 - 1$, and $y = 3 \sin(\pi t)$.

6. Consider the surface $x^2yz + xy - y^2z^2 = -27$.
   (a) Construct an equation of the tangent plane to the surface at $P(1, 3, 2)$.
   (b) Construct a parametrization of the normal line at $P(1, 3, 2)$.

7. Consider the surface $z = x^3y - x^2y^2$. Construct two different vectors that are normal to this surface at $P(2, 1, 4)$.

8. Let $f(x, y, z) = e^x + \cos(y + z)$. Compute the linearization of f at $P(0, \pi/4, \pi/4)$.

Group Work Problems

1. A closed rectangular box 2 inches long, 0.5 inches wide, 4 inches high, is covered by a coat of paint $\frac{1}{16}$ inches thick. Use a differential to estimate the amount of paint on the box.

2. Let $z = f(x, y) = \ln(4x^2 + y^2)$. *Note: this question is based on a 2016 Math 2550 midterm question.*

   (a) Identify all possible unit direction vectors, $\vec{u}$, that satisfy $D_{\vec{u}} f(1, 1) = 0$.

   (b) Identify all points, if any, on $f(x, y) = \ln(4x^2 + y^2)$, where the tangent plane is parallel to the plane $4x - z = 3$.

Partial Solutions to Worksheet 4 Exercises

1. Your vectors should have unit length, should be perpendicular to the level curve at the points they are drawn, and the gradient vectors should point in the direction of steepest ascent.

2. (a) At any point, the maximum rate of change occurs in the direction of $\vec{u} = \nabla f/|\nabla f|$ and the rate of change in this direction is $D_{\vec{u}} = \nabla f \cdot \vec{u}$.

$$\begin{aligned}
\nabla f &= z\hat{i} + 2y\hat{j} + x\hat{k} \\
\nabla f(1,1,2) &= 2\hat{i} + 2\hat{j} + 1\hat{k} \\
|\nabla f(1,1,2)| &= \sqrt{2^2 + 2^2 + 1^2} = 3 \\
\vec{u} &= \frac{1}{3}(2\hat{i} + 2\hat{j} + 1\hat{k}) \\
D_{\vec{u}}(1,1,2) &= \nabla f(1,1,2) \cdot \vec{u}(1,1,2) = \frac{1}{3}(2^2 + 2^2 + 1^2) = 3
\end{aligned}$$

The maximum rate of change at the point is 3, and the direction in which the rate is maximum is $\vec{u} = \frac{1}{3}(2\hat{i} + 2\hat{j} + 1\hat{k})$.

(b) The chain rule gives

$$\begin{aligned}
\frac{df}{dt} &= \nabla f(\vec{r}) \cdot \vec{r}' \\
&= (z\hat{i} + 2y\hat{j} + x\hat{k}) \cdot (\sin t\hat{i} + \hat{j} + \hat{k}) \\
&= z\sin t + 2y + x \\
&= (t+2)\sin t + 2(t+1) + \cos t
\end{aligned}$$

At the given point, $t = 0$, and $f'(0) = 3$.

(c) The total differential of f is df, and $\Delta f \approx df$,

$$\Delta f \approx df = f_x(1,1,2)dx + f_y(1,1,2)dy + f_z(1,1,2)dz = 2(0) + 2(-0.1) + (+0.1) = -0.1$$

A linear approximation to f at $(1,1,2)$ is $f(1,1,2) - 0.1 = 2 + 1 - 0.1 = 2.9$.

3. First compute the gradient at point P.

$$\begin{aligned}
\nabla f &= \frac{\partial}{\partial x}\Big(z\ln(x/y)\Big)\hat{i} + \frac{\partial}{\partial y}\Big(z\ln(x/y)\Big)\hat{j} + \frac{\partial}{\partial z}\Big(z\ln(x/y)\Big)\hat{k} \\
&= z\Big(\frac{\partial}{\partial x}\ln(x/y)\Big)\hat{i} + z\Big(\frac{\partial}{\partial y}\ln(x/y)\Big)\hat{j} + \ln(x/y)\Big(\frac{\partial}{\partial z}(z)\Big)\hat{k} \\
&= \frac{z}{x/y}\frac{\partial}{\partial x}(x/y)\hat{i} + \frac{z}{x/y}\frac{\partial}{\partial y}(x/y)\hat{j} + \ln(x/y)\hat{k} \\
&= \frac{z}{x}\hat{i} - \frac{z}{y}\hat{j} + \ln(x/y)\hat{k} \\
\nabla f(1,1,2) &= 2\hat{i} - 2\hat{j} + 0\hat{k}
\end{aligned}$$

Let the unit vector pointing from $P(1,1,2)$ to $Q(2,2,1)$ be $\vec{u}$.

$$\vec{u} = \frac{1}{|\vec{u}|}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The desired directional derivative is the dot product $\nabla f \cdot \vec{u}$.

$$\nabla f(1,1,2) \cdot \vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Therefore, the directional derivative, at the point $(1,1,2)$, in the direction pointing towards $(2,2,1)$, is zero. Geometrically, this means that the value of the function f is not changing in the specified direction.

4. Use $V = \pi R^2 H$ and use the chain rule.

$$\begin{aligned}
V &= \pi R^2 H \\
\frac{\partial V}{\partial t} &= \frac{\partial V}{\partial R}\frac{dR}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt} \\
&= \frac{\partial(\pi R^2 H)}{\partial R}(-2) + \frac{\partial(\pi R^2 H)}{\partial H}(3) \\
&= 2\pi RH(-2) + (\pi R^2)(3) \\
&= -4\pi RH + 3\pi R^2
\end{aligned}$$

When $R = 10$ and $H = 100$, we have

$$\frac{\partial V}{\partial t} = -4\pi \cdot 10 \cdot 100 + 3\pi(10)^2 = -4000\pi + 300\pi = -3700\pi.$$

5. We can approach this in two different ways. We can use the chain rule, as follows.

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\
&= 2x \cdot 2t + (-2y)(3\pi\cos(\pi t)) \\
&= 4t(t^2-1) - 6\sin(\pi t) \cdot 3\pi\cos(\pi t) \\
&= 4t(t^2-1) - 18\pi\cos(\pi t)\sin(\pi t)
\end{aligned}$$

We could also substitute our known values for x and y first, and then differentiate.

$$\begin{aligned}
\frac{du}{dt} = \frac{\partial}{\partial t}(x^2 - y^2) &= \frac{\partial}{\partial t}\Big((t^2-1)^2 - (3\sin(\pi t))^2\Big) \\
&= 2(t^2-1)(2t) - 6\pi\sin(\pi t) \cdot 3\cos(\pi t) \\
&= 4t(t^2-1) - 18\pi\cos(\pi t)\sin(\pi t)
\end{aligned}$$

6. Let $F(x,y,z) = x^2yz + xy - y^2z^2$. A vector that is perpendicular to this surface at $(1,3,2)$ is $\nabla F(1,3,2)$.

$$\nabla F(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x}F \\ \frac{\partial}{\partial y}F \\ \frac{\partial}{\partial z}F \end{bmatrix} = \begin{bmatrix} 2xyz + y \\ x^2z + x - 2yz^2 \\ x^2y - 2y^2z \end{bmatrix} \Rightarrow \nabla F(1,3,2) = \begin{bmatrix} 15 \\ -21 \\ -33 \end{bmatrix}$$

(a) We now have a vector that is normal to the surface at $(1,3,2)$. The dot product between this vector, and any vector in the plane, is going to be zero.

$$0 = \nabla F(1,3,2) \cdot \begin{bmatrix} x-1 \\ y-3 \\ z-2 \end{bmatrix} = \begin{bmatrix} 15 \\ -21 \\ -33 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-3 \\ z-2 \end{bmatrix}$$

Thus, the tangent plane is given by $15(x-1) - 21(y-3) - 33(z-2) = 0$, which simplifies to $15x - 21y - 33z = -114$.

(b) The normal line is given by the parametric equations

$$x = 1 + 15t, \quad y = 3 - 21t, \quad z = 2 - 33t.$$

7. Let $F(x,y,z) = x^3y - x^2y^2 - z$. Then the surface z has a normal vector given by the gradient ∇F.

$$\nabla F(x,y,z) = \begin{bmatrix} 3x^2y - 2xy^2 \\ x^3 - 2x^2y \\ -1 \end{bmatrix}, \quad \nabla F(2,1,4) = \begin{bmatrix} 3(2)^2(1) - 2(2)(1)^2 \\ 2^3 - 2(2)^2(1) \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

   A vector that is normal to the surface is $[8, 0, -1]$. Another normal vector is $[-8, 0, 1]$.

8. This is Problem 43c in section 14.6. Below is the solution from the solutions manual.

$$f(0, \tfrac{\pi}{4}, \tfrac{\pi}{4}) = 1,\ f_x(0, \tfrac{\pi}{4}, \tfrac{\pi}{4}) = 1,\ \ f_y(0, \tfrac{\pi}{4}, \tfrac{\pi}{4}) = -1,\ f_z(0, \tfrac{\pi}{4}, \tfrac{\pi}{4}) = -1,$$
$$\Rightarrow L(x,y,z) = 1 + 1(x-0) - 1\left(y - \tfrac{\pi}{4}\right) - 1(z - \tfrac{\pi}{4}) = x - y - z + \tfrac{\pi}{2} + 1$$

Group Work Problems

1. $V = xyz$, $x = 2$, $y = 1/2$, $z = 4$. Each of the lengths increases by $2\frac{1}{16}$, so $\Delta x = \Delta y = \Delta z = 2/16$.

$$\begin{aligned} \Delta V \approx dV &= yz\Delta x + xz\Delta y + xy\Delta z \\ &= (2 + 8 + 1)(2/16) = 22/16 \text{ in}^3 \end{aligned}$$

2. Screen captures of handwritten solutions to this problem are below.

let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $|\vec{u}| = 1$.

We need a, b such that

$$0 = \nabla f \cdot \vec{u} \,\Big|_{(1,1)}$$
$$= \frac{1}{4x^2 + y^2} \begin{bmatrix} 8x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \Bigg|_{(1,1)}$$
$$= 8a + 2b$$

$\Rightarrow b = -4a$

But $|\vec{u}| = 1$, so $(-4a)^2 + a^2 = 1$, $a = \pm 1/\sqrt{17}$

$$\Rightarrow \vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \pm 1/\sqrt{17} \\ \mp 4/\sqrt{17} \end{bmatrix}$$

tangent plane to surface $f = \ln(4x^2 + y^2)$ has normal vector ∇F, $F = \ln(4x^2+y^2) - z$, so $\boxed{1}$

$$\nabla F = \frac{1}{4x^2+y^2}\begin{bmatrix} 8x \\ 2y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boxed{1}$$

Plane $4x - z = 3$ has normal vector $\vec{n} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$ $\boxed{1}$
(by inspection)

We want points where ∇F parallel to $\vec{n}$.

$\boxed{1}$ for either (A) or (B)

can use either: $\begin{cases} \nabla F \times \vec{n} = 0, & \text{(A)} \\ \nabla F = \lambda \vec{n}, \quad \lambda = \text{constant} & \text{(B)} \end{cases}$

using (B):

$$\frac{1}{4x^2+y^2}\begin{bmatrix} 8x \\ 2y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow y = 0, \quad \lambda = +1, \quad \left.\begin{aligned} \frac{8x}{4x^2+(0)^2} - 0 &= (+1)\,4 \\ \frac{2}{x} &= +4 \\ x &= +1/2 \end{aligned}\right\} \qquad \boxed{1}$$

$\Rightarrow$ at $(+\frac{1}{2}, 0)$ (sufficient for full marks) $\boxed{1}$

NOTE: if you like: $z = \ln\left(4(\tfrac{-1}{2})^2 + (0)^2\right) = \ln(1) = 0$

$\Rightarrow$ point is $(-\frac{1}{2}, 0, 0)$

NOTE: f not defined at $(0,0)$, so $(0,0)$ is not a solution.

3

V Worksheet 5

Sections from Thomas 13^{th} edition: 14.7, 14.8, 14.9

Exercises

1. Find the dimensions of a rectangular box of maximum volume such that the sum of its 12 lengths is a constant L.

2. Calculate the extreme values of the function $f(x, y) = x^2 + 4y^2 + x - 2y$ on the closed region bounded by $x^2 + 4y^2 = 4$.

3. Consider the function $f(x, y) = 3xy - x^3 - y^3$. Find the points where the tangent plane is horizontal, find the critical points of $f(x, y)$, and classify the critical points as min, max, or saddle points.

4. 14.9.4: Use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations to f near the origin.
$$f(x, y) = \sin(x)\cos(y)$$

Group Work Exercises

1. Use Lagrange Multipliers to calculate the dimensions of an aluminum can in the shape of a cylinder, whose volume is V, and whose surface area is minimum.

2. Problem 41 in section 14.7: A flat circular plate lies in the region $x^2 + y^2 \leq 1$. The plate, including the boundary, is heated so that the temperature at the point (x, y) is $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and coldest points on the plate.

Partial Solutions

1. *Find the dimensions of a rectangular box of maximum volume such that the sum of its 12 lengths is a constant L.*

   Letting the dimensions be a, b, and c, then $V = abc$. To incorporate the length constraint, we will eliminate c by using $4a + 4b + 4c = L$, or $c = L/4 - a - b$. The volume is

$$\begin{aligned} V &= abc = ab(L/4 - a - b) = abL/4 - a^2b - ab^2 \\ V_a &= bL/4 - 2ab - b^2 = 0 \Rightarrow 2a + b = L/4 \\ V_b &= aL/4 - a^2 - 2ab = 0 \Rightarrow 2b + a = L/4 \end{aligned}$$

   Solving these two questions yields $a = b = L/12$. Not surprisingly, $c = L/12$. From the geometrical nature of this problem, this critical point corresponds to a maximum. The rectangular box is a cube with sides of length $L/12$.

   Another approach to this problem would be to use Lagrange Multipliers.

2. Setting the gradient to the zero vector yields the critical points.

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x+1 \\ 8y-2 \end{bmatrix} \quad \Rightarrow \quad (x, y) = (-0.5, 0.25)$$

   The second derivative test yields that this is a minimum, and $f(-0.5, 0.25) = -0.5$.

   Now we need to see if there are any extrema on the boundary. We can use Lagrange Multipliers, or we can use a parametric representation of the curve that represents the boundary. With a parametric approach, we can set

$$x = 2\cos t, \quad y = \sin t$$

   At this point, we can either use the chain rule, or substitute our parametric representation into f and then differentiate. Either approach is fine, but using substitution, $f(x, y) = f(x(t), y(t)) = 2^2\cos^2 t + 4\sin^2 t + 2\cos t - 2\sin t = 4 + 2(\cos(t) - \sin(t))$. Then

$$\begin{aligned} f'(t) &= 0 + 2(-\sin(t) - \cos(t)) \quad \Rightarrow \quad \cos(t) = -\sin(t) \quad \Rightarrow \quad t = \frac{3\pi}{4}, \frac{7\pi}{4} \\ f(\frac{3\pi}{4}) &= 4 - 2\sqrt{2} \\ f(\frac{7\pi}{4}) &= 4 + 2\sqrt{2} \end{aligned}$$

   Putting everything together, the absolute max is $f(\frac{7\pi}{4}) = 4 + 2\sqrt{2}$, absolute min is $f(-0.5, 0.25) = -0.5$.

3. The tangent plane is horizontal at points where $\nabla f(x, y)$ is the zero vector.

$$\nabla f = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{bmatrix}$$

   The gradient vector has zero magnitude when

$$\begin{aligned} 0 &= 3y - 3x^2 \\ 0 &= 3x - 3y^2 \end{aligned}$$

   Rearranging these equations yields the two curves $y = x^2$ and $x = y^2$. These curves intersect at two points, $(0, 0)$, and $(1, 1)$. These are the only two points where the tangent plane is horizontal.

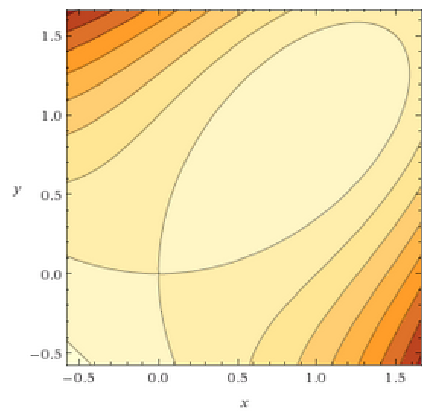
Points where the tangent plane is horizontal correspond to critical points. So the two points, $(0,0)$ and $(1,1)$, **could** indicate local minima/maxima, or they could located a saddle points. We use the second derivative test to tell us if they are.

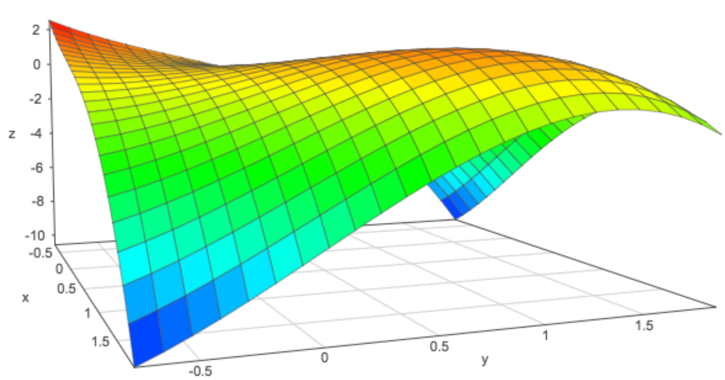
$$D = f_{xx}f_{yy} - f_{xy}^2 = (-6x)(-6y) - (3)(3) = 36xy - 9$$

At $(0,0)$, D is negative, so we have a saddle at $(0,0)$.

At $(1,1)$, $D = 36 - 9 = 27$, and $f_{xx}(1,1) = -6$. Since D is positive and $f_{xx}(1,1)$ is negative, we have a local maximum at $(1,1)$.

In case it helps, shown below are contour and surface plots of $f(x,y) = 3xy - x^3 - y^3$.





4. 14.9.4: Below is a screen capture from a solutions manual.

$f(x, y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y$, $f_y = -\sin x \sin y$, $f_{xx} = -\sin x \cos y$, $f_{xy} = -\cos x \sin y$,

$f_{yy} = -\sin x \cos y \;\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$

$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right) = x$, quadratic approximation;

$f_{xxx} = -\cos x \cos y$, $f_{xxy} = \sin x \sin y$, $f_{xyy} = -\cos x \cos y$, $f_{yyy} = \sin x \sin y$

$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right]$

$= x + \frac{1}{6}\left[x^3 \cdot (-1) + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right] = x - \frac{1}{6}\left(x^3 + 3xy^2\right)$, cubic approximation

Group Work Exercises

1. Let the radius and height of the can be r and h. Then $V = \pi r^2 h$, and the surface area, S, is

$$S = 2\pi r h + 2\pi r^2$$

Using Lagrange Multipliers, $g = g(r, h) = \pi r^2 h - V$, we compute the gradients

$$\nabla S = \begin{bmatrix} S_r \\ S_h \end{bmatrix} = \begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix}$$
$$\nabla g = \begin{bmatrix} 2\pi r h \\ \pi r^2 \end{bmatrix}$$

When S is minimum, the gradients are parallel.

$$\nabla S = \lambda \nabla V$$
$$\begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix} = \lambda \begin{bmatrix} 2\pi r h \\ \pi r^2 \end{bmatrix}$$

The second component gives $2\pi r = \lambda \pi r^2$, or $r = 2/\lambda$. The first component gives us

$$\begin{aligned} 2\pi h + 4\pi r &= \lambda 2\pi r h \\ h + 2r &= \lambda r h \\ h + 4/\lambda &= \lambda(2/\lambda)h \\ \lambda h + 4 &= 2\lambda h \\ h &= 4/\lambda = 2r \end{aligned}$$

But $V = \pi r^2 h$, so

$$V = \pi \frac{16}{\lambda^3} \quad \Rightarrow \quad \lambda^3 = \frac{16\pi}{V} \quad \Rightarrow \quad \lambda = \left(\frac{16\pi}{V}\right)^{1/3}$$

Thus, to minimize surface area for fixed V, use the dimensions

$$r = \frac{2}{\lambda} = 2\left(\frac{16\pi}{V}\right)^{-1/3} = \sqrt[3]{\frac{V}{2\pi}}, \quad h = 2r = 2\sqrt[3]{\frac{V}{2\pi}}$$

2. This was based on Problem 41 in section 14.7. Below is the solution from the solutions manual.

$T_x(x, y) = 2x - 1 = 0$ and $T_y(x, y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$; on the boundary $x^2 + y^2 = 1: T(x, y) = -x^2 - x + 2$ for $-1 \le x \le 1 \Rightarrow T'(x, y) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$; $T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}, T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}, T(-1, 0) = 2,$ and $T(1, 0) = 0 \Rightarrow$ the hottest is $\left(2\frac{1}{4}\right)^\circ$ at $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; the coldest is $\left(-\frac{1}{4}\right)^\circ$ at $\left(\frac{1}{2}, 0\right)$.

VI Worksheet 6

Sections from Thomas 13^{th} edition: 15.1, 15.2, 15.3, 15.4

Exercises

1. 15.4 Problem 32: Calculate the area common to the interiors of the cardioids $r = 1+\cos\theta$ and $r = 1-\cos\theta$.
2. (a) Construct a double integral that represents the volume of the solid enclosed by the cylinder $x^2+y^2 = 1$, the planes $z = 1 - y$, $x = 0$, $z = 0$, in the first octant. Use the integration order $dy\,dx$.
   (b) Change the order of integration to the problem in part (a).
3. Calculate the area of region R in the xy-plane for which the integral is maximum.
$$\iint_R (9 - \frac{x^2}{4} - 4y^2)dA$$
4. Evaluate the double integral.
$$\int_0^4 \int_y^4 e^{x^2}\,dx\,dy$$
5. Evaluate the iterated integral by converting to polar coordinates.
$$\int_0^1 \int_y^{\sqrt{2-y^2}} x\,dx\,dy$$
6. Consider the following iterated integral:
$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} x\,dy\,dx$$
   (a) Sketch the region of integration, R.
   (b) Change the order of integration to $dxdy$.
   (c) Use polar coordinates to evaluate $\iint_R x\,dA$.

Group Work Problems

1. (2 marks) Convert to a double integral in polar coordinates.
$$\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} xy\,dy\,dx$$

2. (2 marks) 15.3 Problem 12: Sketch the region bounded by the lines $y = x - 2$, $y = -x$ and the curve $y = \sqrt{x}$. Construct one or two double integrals that together represent the area of the region.

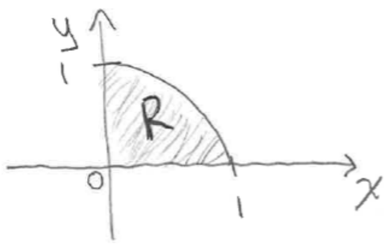
3. (2 bonus marks) Compute the area of the region in the previous problem using one or two double integrals. *These bonus marks are graded for accuracy*.

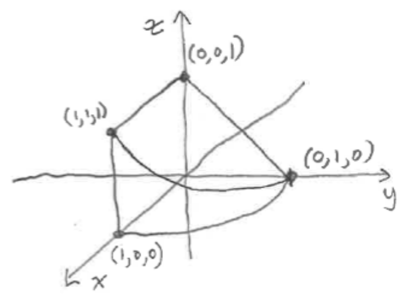
Partial Solutions

1. 15.4 Problem 32: A screen capture from a solutions manual is below.

$$32. \quad A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left(\tfrac{3}{2} - 2\cos\theta + \tfrac{\cos 2\theta}{2}\right)d\theta = \tfrac{3\pi}{2} - 4$$

2. Sketches of region R and the solid are below.





The solid lies under the surface $z = 1 - y$ and above the quarter circle R, with $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$.

$$V = \iint_R f(x,y)dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (1-y)dy\,dx$$

We could also integrate with respect to x first.

$$V = \iint_R f(x,y)dA = \int_0^1 \int_0^{\sqrt{1-y^2}} (1-y)dx\,dy$$

3. The region R is given by all those points (x, y) where $9 - x^2/4 - 4y^2 \le 0$. R cannot have any parts where the integrand is negative, and if there is any domain not part of R where the integrand is positive we could add it to R and increase the value of the integral. R is the region bounded by the ellipse $x^2/4 + 4y^2 = 9$, or

$$\frac{x^2}{6^2} + \frac{2^2y^2}{3^2} = 1$$

The larger semi axis is $a = 6$ and the smaller one is $b = 3/2$. The area is $\pi ab = 9\pi$.

4. The integral $\int e^{x^2}dx$ cannot be expressed in terms of elementary functions. So what can we do to evaluate the double integral? The integration region is bounded by the lines $y = 0$, $x = 4$, and $y = x$. Changing the order of integration, the double integral becomes

$$\begin{aligned}\int_0^4 \int_y^4 e^{x^2}dxdy &= \int_0^4 \int_0^x e^{x^2}dydx \\ &= \int_0^4 ye^{x^2}\Big|_{y=0}^{y=x}dx \\ &= \int_0^4 xe^{x^2}dx \\ &= \frac{e^{x^2}}{2}\Big|_0^4 \\ &= \frac{e^{16}-1}{2}\end{aligned}$$

Changing the order of integration can sometimes make it easier to evaluate certain integrals.

5. The integral

$$\int_0^1 \int_y^{\sqrt{2-y^2}} x\, dx\, dy$$

becomes

$$\int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta)\, r\, dr\, d\theta$$

Thus,

$$\begin{aligned}\int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta)\, r\, dr\, d\theta &= \int_0^{\pi/4} \cos\theta\, d\theta \int_0^{\sqrt{2}} r^2\, dr \\ &= \frac{\sqrt{2}}{2} \int_0^{\sqrt{2}} r^2\, dr \\ &= \frac{\sqrt{2}}{2}\, \frac{(\sqrt{2})^3}{3} \\ &= \frac{2}{3}\end{aligned}$$

6. Handwritten solutions to this problem are in a screen capture below.

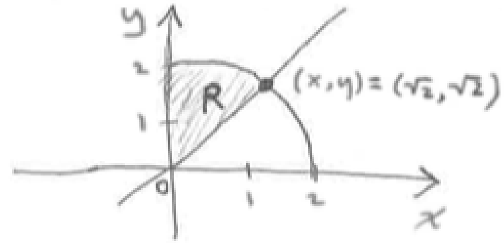
(a) Draw the region R of integration in the plane.

$x \in [0, \sqrt{2}]$

$y \in [x, \sqrt{4-x^2}]$

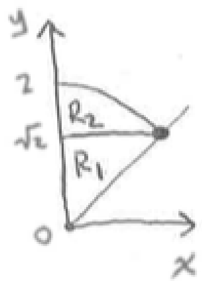
if $y = \sqrt{4-x^2}$

then $y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$



$$R = \{(x,y) \mid 0 \le x \le \sqrt{2},\ x \le y \le \sqrt{4-x^2}\}$$

(b) Find iterated integral(s) representing the same double integral, i.e. $\iint_R x\, dA$, but with the order of integration switched to $dx\,dy$.



$$\underbrace{\int_0^{\sqrt{2}} \int_0^{y} x\, dx\, dy}_{R_1} + \underbrace{\int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-y^2}} x\, dx\, dy}_{R_2}$$

(c) Use polar coordinates to evaluate $\iint_R x\, dA$.

$$\iint_R x\, dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^2 r^2\, dr$$

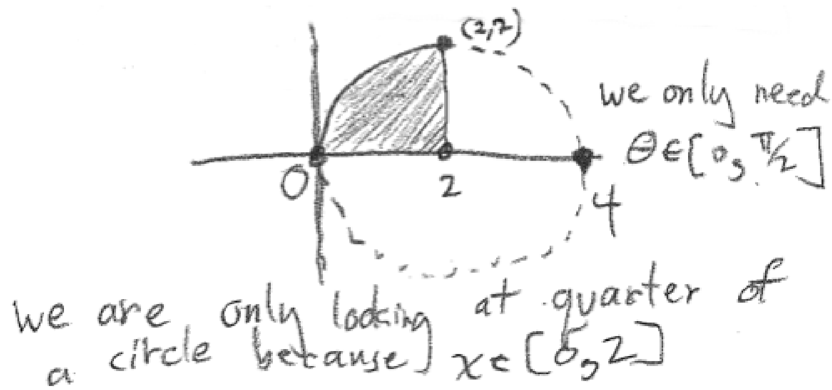
$$= \sin\theta \Big|_{\pi/4}^{\pi/2} \cdot \frac{r^3}{3}\Big|_0^2 = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{8}{3}$$

Group Work Problems

1. We are given that $x \in [0, 2]$ and $y \in [0, \sqrt{4-(x-2)^2}]$. To understand the shape of the region of integration, let's rearrange our limits for y.

$$\begin{aligned} 0 \leq y &\leq \sqrt{4-(x-2)^2} \\ y^2 &\leq 4-(x-2)^2 \\ (x-2)^2 + y^2 &\leq 2^2 \end{aligned}$$

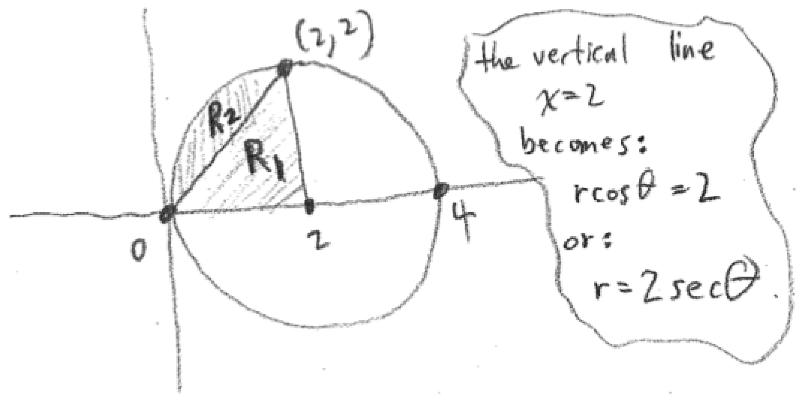
The region is bounded by the circle of radius 2, centered at $(2, 0)$.



Now convert to polar.

$$\begin{aligned} (x-2)^2 + y^2 &\leq 2^2 \\ y^2 + x^2 &\leq 4x \\ r^2 &\leq 4r\cos\theta \\ r &\leq 4\cos\theta \end{aligned}$$

Note that if $y \geq 0$, then $\theta \in [0, \pi]$. Divide the quarter circle into two regions, R_1 and R_2.



R_1: $\theta \in [0, \frac{\pi}{4}]$, $r \in [0, \frac{2}{\cos\theta}]$

R_2: $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$, $r \in [0, 4\cos\theta]$

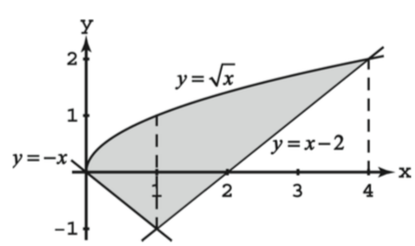
Thus, the double integral in Cartesian becomes two double integrals in polar.

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} xy\,dydx &= \iint_{R_1} dA + \iint_{R_2} dA \\ &= \int_0^{\pi/4} \int_0^{2\sec\theta} (r\cos\theta)(r\sin\theta)\,rdrd\theta + \int_{\pi/4}^{\pi/2} \int_0^{4\cos\theta} (r\cos\theta)(r\sin\theta)\,rdrd\theta\end{aligned}$$

Note: the question above was taken directly from a 2014 Math 2401 Quiz.

2. 15.3 Problem 12: A screen capture from a solutions manual is below.

12. $$\begin{aligned}&\int_0^1\int_{-x}^{\sqrt{x}} 1\,dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} 1\,dy\,dx \\ &= \int_0^1 [y]_{-x}^{\sqrt{x}}\,dx + \int_1^4 [y]_{x-2}^{\sqrt{x}}\,dx \\ &= \int_0^1 \left(\sqrt{x}+x\right)dx + \int_1^4 \left(\sqrt{x}-x+2\right)dx \\ &= \left[\tfrac{2}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2 + 2x\right]_1^4 = \tfrac{13}{3}\end{aligned}$$



VII Worksheet 7

Sections from Thomas 13^{th} edition: 15.5, 15.6, 15.7, 15.8

Exercises

1. 15.5 Problem 30: Find the volume of the region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$.
2. A solid is bounded below by $z = \sqrt{x^2 + y^2}$ and above by $z = 1$. The density of the solid at each point P in the solid is proportional the distance from P to the z-axis.
   (a) Calculate the mass of the solid.
   (b) Calculate the center of mass of the solid.
3. Use a change of variables to evaluate $\iint_\Omega (x + y) dA$, where Ω is the region bounded by
$$x - y = 0, \quad x - y = \pi, \quad x + 2y = 0, \quad x + 2y = \pi/2$$
4. Construct a triple integral that represents the volume of the solid bounded above by $z = 1$ and below by $z = \sqrt{x^2 + y^2}$.
5. Construct an integral that represents the volume of the region bounded by $x^2 + y = 1$ and $z^2 + y = 1$, in the first octant.
6. Set up the triple integral in at least two other ways by changing the integration order.
$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

Group Work Problems

1. (2 marks) Calculate the volume of the ice cream cone, bounded below by $z = \sqrt{3x^2 + 3y^2}$ and bounded above by $x^2 + y^2 + z^2 = 1$.
2. (2 marks) A single wedge is cut from a spherical ball of radius R by two planes that meet in a diameter. The angle between the planes is $\alpha \in (0, \pi/2)$. Use spherical coordinates to construct a triple integral that represents the volume of the wedge.
3. (bonus 2 marks) In the previous question, what is the volume of the solid that remains? *This bonus question is graded for accuracy.*

Partial Solutions

1. 15.5 Problem 30: The solution for this problem is as shown in the screen capture from a solutions manual below.

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dy\, dx = \int_0^2 \int_0^{4-x^2} \left(4-x^2-y\right) dy\, dx = \int_0^2 \left[\left(4-x^2\right)^2 - \frac{1}{2}\left(4-x^2\right)^2\right] dx$

$= \frac{1}{2}\int_0^2 \left(4-x^2\right)^2 dx = \int_0^2 \left(8-4x^2+\frac{x^4}{2}\right) dx = \frac{128}{15}$

2. (a) The distance from any point in the solid to the z-axis is r. The density function is $\delta(x, y) = kr$, where k is a constant of proportionality.

$$M = \int_0^{2\pi} \int_0^1 \int_r^1 kr^2 dz\, dr\, d\theta = 2\pi k \int_0^1 r^2(1-r)\, dr = 2\pi k\left(\frac{1}{3} - \frac{1}{4}\right) = k\pi/6$$

   (b) $\bar{x} = \bar{y} = 0$ by symmetry.

$$M\bar{z} = \int_0^1 \int_0^{2\pi} \int_r^1 r^2 zk\, dz\, dr\, d\theta = 2\pi k \int_0^1 r^2(1-r^2)\, dr = 2\pi k/15 \Rightarrow \bar{z} = 4/5$$

3. Let $u = x - y$ and $v = x + 2y$. Then $x = (2u + v)/3$, $y = (v - u)/3$, and $J = 1/3$.

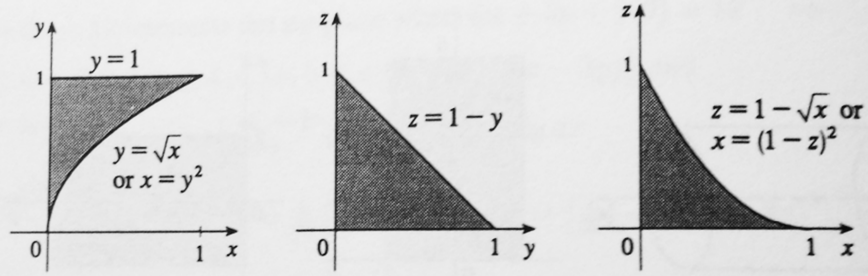
$$\iint_\Omega (x+y) dA = \frac{1}{9}\int_0^\pi \int_0^{\pi/2} (u + 2v)\, dv\, du = \frac{\pi^3}{18}$$

4. The solid is a cone with vertex at the origin.

$$volume = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 dzdydx$$

5. The volume can be represented with the triple integral $\displaystyle\int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-y}} dzdydx$

6. Below is a screen capture that gives all of the different ways the integral could be constructed in Cartesian.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\begin{aligned}
\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz.
\end{aligned}$$

Group Work Problems

1. Using cylindrical coordinates, we obtain

$$V = \int_0^{2\pi}\int_0^{1/2}\int_{r\sqrt{3}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \frac{\pi(2-\sqrt{3})}{3}$$

2. If we position the planes so that they meet along the z-axis, assume that the planes create only one wedge, and one of the planes is the xz-plane, then the volume of the wedge is:

$$V_{wedge} = \int_0^{\alpha}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

   In case it helps, the solid that is removed looks a bit like one slice of an orange.

3. The volume of the wedge is:

$$V_{wedge} = \int_0^{\alpha}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{3}\alpha R^3$$

   The volume of the solid that remains is

$$V_{solid} = \frac{4}{3}\pi R^3 - \frac{2}{3}\alpha R^3 = \frac{2R^3}{3}(2\pi - \alpha)$$

VIII Additional Review Problems for Midterm 1

Sections from Thomas 13^{th} edition: 12.4, 12.5, 12.6, 13.1, 13.2, 13.3, 13.4

Exercises

1. Construct a vector function, $\vec{r}(t)$, whose curvature is $\frac{1}{4}$ for $t \geq 0$, and $\vec{r}(0) = 5\hat{j}$.
2. Construct a parametric representation of the curve that lies in the intersection of the surface $x^2 + 2y^2 = z$ and the plane $x - y = 5$. Draw a rough sketch of the curve.
3. A ball rolls off a table 1 meter high with a speed of 0.5 m/s.
   (a) At what speed does the ball strike the floor?
   (b) Where does the ball strike the floor?
4. Consider the surface $x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$.
   (a) Name the surface.
   (b) Draw a rough sketch of the surface. Do not forget to label the axes.
5. Do the following lines intersect? If so, where? If not, calculate the distance between the two lines.
$$\vec{r}_1(t) = 3\hat{\imath} + \hat{\jmath} + 5\hat{k} + t(\hat{\imath} - \hat{\jmath} + 2\hat{k})$$
$$\vec{r}_2(s) = \hat{\imath} + 4\hat{\jmath} + 2\hat{k} + s(\hat{\jmath} + \hat{k})$$
6. Problem 25 in 13.1: A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point $(2, 2)$.
7. Problem 62 in Chapter 12 practice Exercises: a parallelogram has vertices $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Vectors $\overrightarrow{BA}$ and $\overrightarrow{CD}$ point in the same direction.
   (a) Find the coordinates of D.
   (b) Find the area of the parallelogram.
8. Let C be the curve given by the intersection of $x^2 + y^2 + z^2 = 25$ and $x + y + z = 0$. Calculate the length of C. *Hint: no integration is necessary.*
9. Compute the distance between the skew lines.
$$L_1 : x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - 2t$$
$$L_2 : x = 2u, \quad y = 2 - u, \quad z = -3 + 2u$$
10. Surface S consists of the set of points whose distance from the x-axis are equal to $\sqrt{1 - x}$. Construct an equation that represents S, name the surface, and create a rough sketch of it.
11. Starting from the point $P(0, 1, 1)$, an object moves with velocity $\vec{v}(t) = \langle \pi \sin(\pi t), -1, \pi \cos(\pi t) \rangle$, for $0 \leq t \leq 2$.
    (a) (6 points) Obtain the curve, $\vec{r}(t)$, that gives the object's position.

(b) (4 points) On one graph, sketch $\vec{r}(t)$, indicate the direction of motion, and sketch the velocity vector at time $t = 0$.

12. The position of an object is given by $\vec{r}(t) = \langle 2\cos(t), \sqrt{2}\sin(t), \sqrt{2}\sin(t)\rangle$ for $t \geq 0$.

    (a) Compute the unit tangent vector $\mathbf{T}(t)$.

    (b) Compute the unit normal vector $\mathbf{N}(t)$.

    (c) At what times, if any, is the curvature equal to 2?

    (d) When has the object travelled a distance of 3 units?

Solutions

Partial solutions to all of the above problems will be posted shortly after the lecture prior to their midterm. Students are encouraged to work on all of the above problems before comparing their work to the solutions.

Partial Solutions to Additional Review Problems for Midterm 1

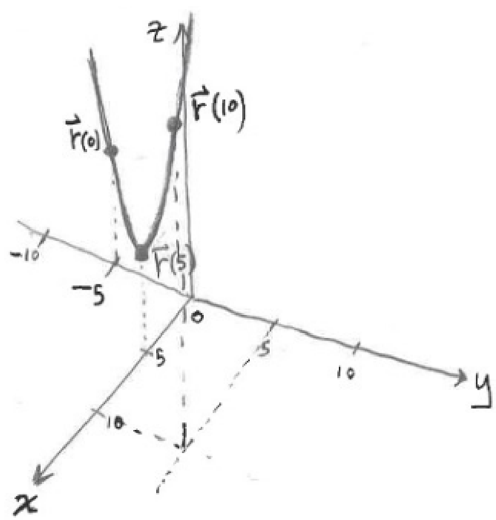
1. Circles have constant curvature equal to the inverse of their radius. The curve $x^2 + (y-1)^2 = 4^2$ is a circle with radius 4 and has curvature $\frac{1}{4}$ for all points on the curve. This curve also passes through the point $(0, 5)$. The parametrization $x = 4\sin(t)$, $y = 4\cos(t) + 1$ satisfies the equation of this circle. The parametrization also yields a vector function $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$ that has curvature $\frac{1}{4}$ for all t, and that satisfies $\vec{r}(0) = 5\hat{\jmath}$.

2. Let $x = t$, then $y = t - 5$, and $z = t^2 + 2(t-5)^2$. A vector function that represents the intersection of the surfaces is

$$\vec{r}(t) = \begin{bmatrix} t \\ t-5 \\ t^2 + 2(t-5)^2 \end{bmatrix}$$

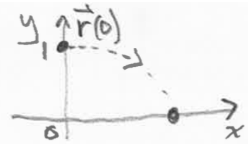
To draw a rough sketch of a curve, it is often helpful to identify a few points on the curve and then connect the dots in the sketch.

$$\vec{r}(0) = \begin{bmatrix} 0 \\ -5 \\ 50 \end{bmatrix}, \quad \vec{r}(5) = \begin{bmatrix} 5 \\ 0 \\ 25 \end{bmatrix}, \quad \vec{r}(10) = \begin{bmatrix} 10 \\ 5 \\ 150 \end{bmatrix}$$



3. Below is a screen capture of handwritten notes.

R05 **1) Ball Rolling off of a Table (Projectile Motion, 13.2)**



5 A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

a) What speed does the ball strike the floor?

b) Where does the ball strike the floor?

a) acceleration $= \vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}$, so velocity $= \vec{v}(t) = \begin{bmatrix} c_1 \\ -gt + c_2 \end{bmatrix}$. But $\vec{v}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$,

so $c_1 = \frac{1}{2}$, $c_2 = 0$. Thus, $\vec{v}(t) = \begin{bmatrix} 1/2 \\ -gt \end{bmatrix}$. Position $= \vec{r}(t) = \begin{bmatrix} \frac{1}{2}t + d_1 \\ -gt^2/2 + d_2 \end{bmatrix}$.

But $\vec{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $d_1 = 0$, $d_2 = 1$.

Ball hits floor when $0 = -gt^2/2 + 1$, or $t = \sqrt{2/g}$

$\Rightarrow \left|\vec{v}\left(\sqrt{2/g}\right)\right| = \sqrt{(1/2)^2 + \left(g\left(\sqrt{2/g}\right)\right)^2} \approx 4.455\ldots$

b) $\vec{r}\left(\sqrt{\frac{2}{g}}\right) = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{g}} \\ -g\left(\sqrt{\frac{2}{g}}\right)^2/2 + 1 \end{bmatrix} \approx \begin{bmatrix} 0.22576\ldots \\ 0 \end{bmatrix}$

$\Rightarrow$ at point $(0.2, 0)$

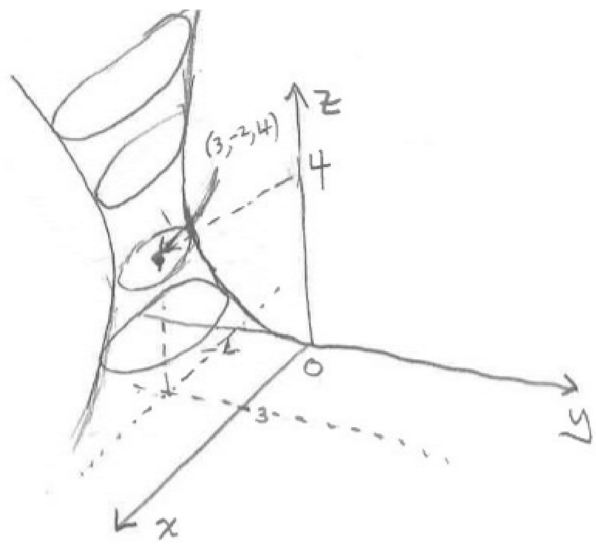
4. First express the given equation for the surface into standard form so that we can more easily identify and sketch it.

$$\begin{aligned}
(x^2 - 6x + 9 - 9) + (y^2 + 4y + 4 - 4) - (z^2 - 8z + 16 - 16) &= 4 \\
(x-3)^2 - 9 + (y+2)^2 - 4 - (z-4)^2 + 16 &= 4 \\
(x-3)^2 + (y+2)^2 - (z-4)^2 &= 1
\end{aligned}$$

The surface is centered on the point $(3, -2, 4)$.

(a) By inspection, the surface is a hyperboloid of one sheet.

(b)



5. Set $\vec{r}_1 = \vec{r}_2$ yields three equations in two unknowns. Solving gives $t = -2$ and $s = -1$. Substitution gives the point of intersection, which is $(1, 3, 1)$.

6. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\hat{\imath}$ component. Implicit differentiation of $y^2 = 2x$ leads to $2y\frac{dy}{dx} = 2$, so at $(2, 2)$, $\frac{dy}{dx} = 1/2$. The tangent vector at $(2, 2)$ points in the direction of the vector $\hat{\imath} + 0.5\hat{\jmath}$. A unit vector in this direction is

$$\frac{1}{\sqrt{1 + 1/4}}\left(\hat{\imath} + 0.5\hat{\jmath}\right)$$

The desired velocity vector is

$$\frac{5}{\sqrt{1 + 1/4}}\left(\hat{\imath} + 0.5\hat{\jmath}\right)$$

7. (a) Here are two approaches you can use to solve this problem.
   i. The line through A and B is $x = 1 + t, y = -t, z = -1 + 5t$; the line through C and D must be parallel and is $L_1 : x = 1+t, y = 2-t, z = 3+5t$. The line through B and C is $x = 1, y = 2+2s, z = 3 + 4s$; the line through A and D must be parallel and is $L_2 : x = 2, y = -1 + 2s, z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$, where $t = 1$ and $s = 1$.
   ii. $\vec{BA} = \hat{\imath} - \hat{\jmath} + 5\hat{k}$, vectors $\vec{BA}$ and $\vec{CD}$ are parallel, so D is located at $D(1 + 1, 2 - 1, 3 + 5)$, or $D(2, 1, 8)$.

   (b) The area is $|(2j + 4k) \times (i - j + 5k)| = 6\sqrt{6} = \sqrt{216}$

8. The curve C is a great circle on a sphere of radius 5. The length is the circumference of a circle of radius 5, which is 10π.

9. The vector $\vec{v}_1 = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ is parallel to L_1, and $\vec{v}_2 = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ is parallel to L_2. Thus $\vec{u} = \vec{v}_1 \times \vec{v}_2 = 14\hat{\imath} - 8\hat{\jmath} - 5\hat{k}$ is perpendicular to both lines. Point $P(1, -2, 4)$ is on line L_1, and point $S(0, 3, -3)$ is on line L_2. The length of projection of $\overrightarrow{PS}$ onto $\vec{u}$ gives the desired distance.

$$\left|\text{proj}_{\vec{u}}\overrightarrow{PS}\right| = \left|\frac{\overrightarrow{PS}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\right| = \frac{\left|\overrightarrow{PS}\cdot\vec{u}\right|}{|\vec{u}|} = \frac{|-14 - 40 + 35|}{\sqrt{5^2 + 8^2 + 14^2}} = \frac{19}{\sqrt{285}}$$

10. The distance from any point (x, y, z) to the x-axis is $\sqrt{z^2 + y^2}$.

$$\sqrt{z^2+y^2} = \sqrt{1-x}$$
$$x = 1 - z^2 - y^2$$

By inspection, the surface is a paraboloid with vertex at the point $(1, 0, 0)$. The paraboloid opens over the negative x-axis.

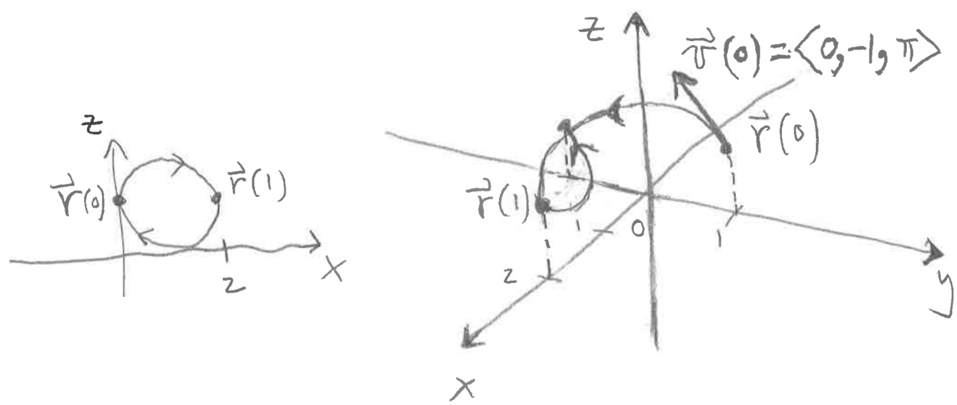
11. (a)

$$\vec{r} = \int \vec{v}dt = \langle -\cos(\pi t), -t, \sin(\pi t)\rangle + \vec{C}$$
$$\vec{r}(0) = \langle 0, 1, 1\rangle = \langle -1, 0, 0\rangle + \vec{C}$$
$$\vec{C} = \langle 1, 1, 1\rangle$$
$$\vec{r} = \langle 1 - \cos(\pi t), 1 - t, 1 + \sin(\pi t)\rangle$$

(b) To draw a rough sketch of a curve, it is often helpful to identify a few points on the curve and then connect the dots in the sketch.

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{r}(1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{r}(2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

It can also help to sketch the curve in one of the coordinate planes.



12. (a)

$$\vec{v} = \langle -2\sin(t), \sqrt{2}\cos(t), \sqrt{2}\cos(t)\rangle$$
$$|\vec{v}| = \sqrt{(-2\sin(t))^2 + (\sqrt{2}\cos(t))^2 + (\sqrt{2}\cos(t))^2} = 2$$
$$\vec{T} = \vec{v}/|\vec{v}| = \frac{1}{2}\langle -2\sin(t), \sqrt{2}\cos(t), \sqrt{2}\cos(t)\rangle$$

(b)

$$\vec{T}\,' = \frac{1}{2}\langle -2\cos(t), -\sqrt{2}\sin(t), -\sqrt{2}\sin(t)\rangle$$
$$|\vec{T}\,'| = 1$$
$$\vec{N} = \vec{T}\,'/|\vec{T}\,'| = \frac{1}{2}\langle -2\cos(t), -\sqrt{2}\sin(t), -\sqrt{2}\sin(t)\rangle$$

(c) Using results from the previous parts of this question,

$$\kappa(t) = \frac{1}{|\vec{v}|}\left|\vec{T}\,'\right| = \frac{1}{2}\,1 = \frac{1}{2}$$

The curvature is constant for all t and is never equal to 2.

(d) Using results from the previous parts of this question,

$$3 = \int_0^t |\vec{v}(\tau)|d\tau = \int_0^t 2d\tau = 2t$$

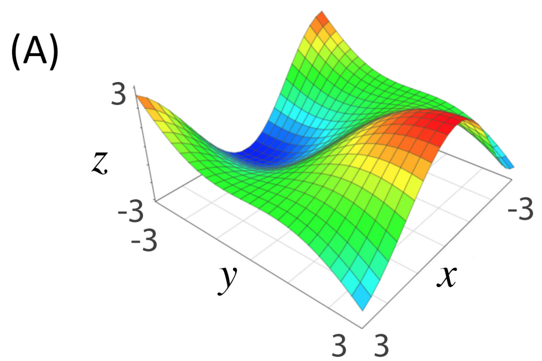
The object has moved a distance of 3 units when $t = 3/2$.

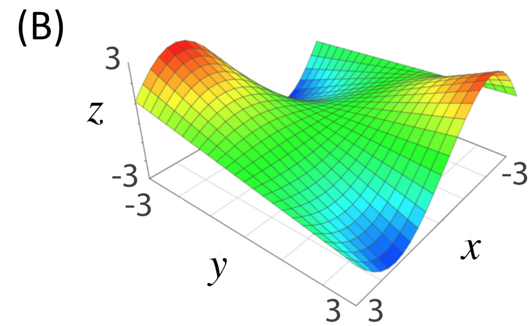
IX Additional Review Problems for Midterm 2

Sections from Thomas 13^{th} edition: 14.1, 14.2, 14.3, 14.4, 14.5, 14.6

Exercises

1. Below is a plot of a function, $f(x, y)$, and $f_x(x, y) = \frac{\partial f}{\partial x}(x, y)$. Which one is which? Identify each plot as either $f(x, y)$, or $f_x(x, y)$, and justify your reasoning.





2. Construct an equation for the line tangent to the curve of intersection of the surfaces $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$ at $P(-1, 1, 2)$.

3. Sketch the level curves for the indicated values of C, if possible.

   (a) $f(x, y) = \frac{\ln y}{x}$, $C = -1, 0, 1$

   (b) $f(x, y) = \frac{x^2}{x^2+y^2}$, $C = 0, \frac{1}{2}, 1, 2$

4. Determine the domain of each function, and indicate whether the domain is open and/or closed.

   (a) $R(x, y) = x^2 + y^2$

   (b) $f(x, y) = \ln(1 - x^2 - y^2)$

   (c) $g(x, y) = \sqrt{1 - x^2 - y^2}$

   (d) $h(x, y) = \dfrac{\sqrt{1 - x^2 - y^2}}{x - y}$

   (e) $u(x, y) = \dfrac{\ln(1 - x)}{x^2 + y^2}$

5. Evaluate the following limits, or show that they do not exist.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$

   (b) $\displaystyle \lim_{(x,y)\to(0,1)} \frac{xy - x}{x^2 + y^2 - 2y + 1}$

6. Construct a function, $f(x, y)$, whose level curves are straight lines that meet at the point $P(0, 1)$.

7. Construct a function, $f(x, y)$, whose level curves, $f(x, y) = C$ are circles centered on the origin, and have radius $\sqrt{e^C}$.

8. Compute the value of $\frac{\partial z}{\partial x}$ if
$$x^2 + y^2 - z^2 = 2x(y+z)$$
defines z as a function two independent variables y and x.

9. Problem 42a in Section 14.4. The lengths, a, b, c of the edges of a rectangular box are changing with time. At a particular time, $a = 1$ m, $b = 2$ m , $c = 3$ m , $da/dt = db/dt = 1$ m/sec, and $dc/dt = -3$ m/sec. At what rates is the volume V and surface area S of the box changing?

10. The temperature of an object, T, at time t, is a function of its density, ρ, and its distance from a point, D.
$$T = f(\rho, D, t)$$
Suppose $t \geq 0$ and
$$\rho(t) = 2 - e^{-t}, \qquad D(t) = \frac{1}{t+1} \ .$$
Suppose also that $t = 1$,
$$\frac{\partial f}{\partial \rho} = 2, \quad \frac{\partial f}{\partial D} = -2, \quad \frac{\partial f}{\partial t} = 1 \ .$$
Is the temperature at $t = 1$ increasing or decreasing?

11. Construct an equation for the plane that is tangent to $z = 4x^2 + y^2$ at $P_0(1,\, 1,\, 5)$. Also construct an equation for the normal line at the point.

12. Identify the directions in which the function increases and decreases most rapidly at $P_0(1,\, 0)$. Then find the derivatives of the function in these directions.
$$f(x,\, y) = x^2 y + e^{xy} \sin y$$

13. Problem 20 from 14.6: By about how much will
$$f(x,\, y,\, z) = e^x \cos yz$$
change as the point $P(x,\, y,\, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

Solutions

Partial solutions to all of the above problems will be posted shortly after the lecture prior to their midterm. Students are encouraged to work on all of the above problems before comparing their work to the solutions.

Partial Solutions

1. Consider the point where $x = -3$, $y = -3$. If (A) were $f(x, y)$, then the rate of change, in the x-direction, is positive, at $x = -3$, $y = -3$. The surface in (B) at this point is positive.

   If (B) were $f(x, y)$, then the rate of change at $x = -3$ and $y = -3$ in the x-direction, is negative, and the value of the surface in (A) at $x = -3$ and $y = -3$ is positive, so (B) cannot be $f(x, y)$.

   So (A) is $f(x, y)$, and (B) is $f_x(x, y)$.

   Still not convinced? Try considering another point, such as $(-3, 3)$. Also, the functions used to create these plots were $f(x, y) = y\cos(x) + \sin(y)$ and $f_x(x, y) = -y\sin(x)$. Try plotting the surfaces in your favorite graphing software on the interval $-3 \le x \le 3$, and $-3 \le y \le 3$.

2. Let $f = z - x^2 - y^2$ and $g = 4x^2 + y^2 + z^2 - 9$. Then the tangent line is perpendicular to both ∇f and ∇g. Vector $\vec{v} = \nabla f \times \nabla g$ is parallel to the desired tangent line.

$$\nabla f(x, y, z) = \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix}, \quad \nabla f(-1, 1, 2) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$
$$\nabla g(x, y, z) = \begin{bmatrix} 8x \\ 2y \\ 2z \end{bmatrix}, \quad \nabla g(-1, 1, 2) = \begin{bmatrix} -8 \\ 2 \\ 4 \end{bmatrix}$$
$$\nabla f(-1, 1, 2) \times \nabla g(-1, 1, 2) = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = \begin{bmatrix} -10 \\ -16 \\ -12 \end{bmatrix}$$

   Parametric vector equations for the tangent line at $(-1, 1, 2)$ are

$$x = -1 - 10t, \quad y = 1 - 16t, \quad z = 2 - 12t.$$

3. (a) To plot the level curves, we set $z = C$, and solve for y:

$$\begin{aligned} C &= \frac{\ln y}{x} \\ Cx &= \ln y \\ y &= e^{cx} \end{aligned}$$

   (b) To plot the level curves, we set $z = C$, and solve for y:

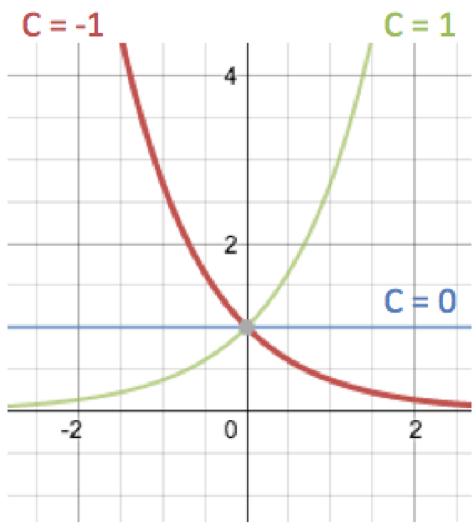
$$\begin{aligned} C &= \frac{x^2}{x^2 + y^2} \\ Cy^2 &= (1 - C)x^2 \end{aligned}$$

   If $C = 0$, then we obtain the level curve $x = 0$. If $C \neq 0$, then

$$y = \pm\sqrt{\frac{1 - C}{C}}x, \quad C \neq 0$$

   For $C = 2$, y is undefined, so the level curve for $C = 2$ does not exist (because the range of f is the closed interval $[0, 1]$. The level curves for the other values are:

$$\begin{aligned} &C = 0, \quad x = 0 \text{ (the vertical line that lies on the } y\text{-axis)} \\ &C = 1/2, y = \pm x \\ &C = 1, \quad y = 0 \end{aligned}$$



4. (a) The domain is $\mathbb{R}^2$, the boundary is empty, and so the the domain is both closed and open, or clopen.
   (b) The domain is the set $D = \{(x, y)|x^2 + y^2 < 1\}$. The domain does not include its boundary, so the domain is open.
   (c) The domain is the set $D = \{(x, y)|x^2 + y^2 \leq 1\}$. The domain includes its entire boundary, so the domain is closed.
   (d) The domain is the set $D = \{(x, y)|x^2 + y^2 \leq 1, x \neq y\}$. The domain does not include its entire boundary, so the domain is not closed. The domain consists of points in its boundary, so the domain is not open. The domain is neither open nor closed.
   (e) The domain is the set $D = \{(x, y)|x^2 + y^2 \neq 0, x < 1\}$. The domain consists entirely of interior points, so the domain is open.
5. (a) Along straight lines that pass through the origin, $y = kx$, and

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2(kx)}{x^4 + (kx)^2} = 0$$

   Along any straight line, the limit is zero. If we approach the limit point along parabolas $y = kx^2$,

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{k}{1 + k^2}$$

   The result depends on k, so the limit does not exist.
   (b) Try evaluating the limit along the y-axis and along $y = x + 1$.
6. The set of straight lines are given by $y = Cx + 1$. Rearranging yields

$$C = \frac{y - 1}{x}, x \neq 0$$

A function that meets the given criteria is

$$f(x, y) = \frac{y - 1}{x}, x \neq 0$$

There are other functions that will also meet the given criteria.

7. The set of circles with radius e^C are given by $x^2+y^2=e^C$. Applying the natural logarithm to both sides of the equation yields
$$\ln(x^2+y^2)=C$$
A function that meets the given criteria is
$$f(x,y)=\ln(x^2+y^2)$$
There are other functions that will also meet the given criteria.

8. You should obtain that $\frac{\partial z}{\partial x}=\frac{x-y-z}{z+x}$.

9. This is Problem 42a in section 14.4. Below is the solution from the solutions manual for both parts (a) and (b). Part (b) involves finding the rate at which the lengths of the diagonals are changing.

$V=abc \Rightarrow \frac{dV}{dt}=\frac{\partial V}{\partial a}\frac{da}{dt}+\frac{\partial V}{\partial b}\frac{db}{dt}+\frac{\partial V}{\partial c}\frac{dc}{dt}=(bc)\frac{da}{dt}+(ac)\frac{db}{dt}+(ab)\frac{dc}{dt}$
$\Rightarrow \left.\frac{dV}{dt}\right|_{a=1,\,b=2,\,c=3}=(2\ m)(3\ m)(1\ m/\text{sec})+(1\ m)(3\ m)(1\ m/\text{sec})+(1\ m)(2\ m)(-3\ m/\text{sec})=3\ m^3/\text{sec}$ and the volume is increasing; $S=2ab+2ac+2bc \Rightarrow \frac{dS}{dt}=\frac{\partial S}{\partial a}\frac{da}{dt}+\frac{\partial S}{\partial b}\frac{db}{dt}+\frac{\partial S}{\partial c}\frac{dc}{dt}=2(b+c)\frac{da}{dt}+2(a+c)\frac{db}{dt}+2(a+b)\frac{dc}{dt}$
$\Rightarrow \left.\frac{dS}{dt}\right|_{a=1,\,b=2,\,c=3}=2(5\ m)(1\ m/\text{sec})+2(4\ m)(1\ m/\text{sec})+2(3\ m)(-3\ m/\text{sec})=0\ m^2/\text{sec}$ and the surface area is not changing; $D=\sqrt{a^2+b^2+c^2} \Rightarrow \frac{dD}{dt}=\frac{\partial D}{\partial a}\frac{da}{dt}+\frac{\partial D}{\partial b}\frac{db}{dt}+\frac{\partial D}{\partial c}\frac{dc}{dt}=\frac{1}{\sqrt{a^2+b^2+c^2}}\left(a\frac{da}{dt}+b\frac{db}{dt}+c\frac{dc}{dt}\right)$
$\Rightarrow \left.\frac{dD}{dt}\right|_{a=1,\,b=2,\,c=3}=\left(\frac{1}{\sqrt{14}\ m}\right)\left[(1\ m)(1\ m/\text{sec})+(2\ m)(1\ m/\text{sec})+(3\ m)(-3\ m/\text{sec})\right]=-\frac{6}{\sqrt{14}}\ m/\text{sec}<0 \Rightarrow$ the diagonals are decreasing in length

10. First we construct an expression for
$$\frac{dT}{dt}$$
in terms of f and t.
$$\frac{dT}{dt}=\frac{\partial f}{\partial \rho}\frac{d\rho}{dt}+\frac{\partial f}{\partial D}\frac{dD}{dt}+\frac{\partial f}{\partial t}\frac{dt}{dt}=e^{-t}\frac{\partial f}{\partial \rho}+\frac{-1}{(t+1)^2}\frac{\partial f}{\partial D}+\frac{\partial f}{\partial t}$$
Next we substitute our given values for the partial derivatives.
$$\begin{aligned}\frac{dT}{dt}&=e^{-t}\frac{\partial f}{\partial \rho}+\frac{-1}{(t+1)^2}\frac{\partial f}{\partial D}+\frac{\partial f}{\partial t}\\&=\frac{1}{e}\,2+\frac{-1}{(1+1)^2}(-2)+1\\&=\frac{3}{2}+\frac{2}{e}\end{aligned}$$
The rate of change of temperature is positive, so the temperature is increasing when $t=1$.

11. *Many thanks to JD Walsh for writing the question and solution for this problem.*
Let $f(x,\,y)=z=4x^2+y^2$. Here, we need only calculate two partial derivatives
$$f_x=8x \qquad\qquad f_y=2y$$

At the point $P_0(x_0,\, y_0,\, z_0) = (1,\, 1,\, 5)$ we have

$$f_x(1,\, 1) = 8 \qquad\qquad f_y(1,\, 1) = 2$$

The equation for the plane tangent to the surface is given by

$$f_x(x_0,\, y_0)(x - x_0) + f_y(x_0,\, y_0)(y - y_0) - (z - z_0) = 0$$

Therefore, we have

$$8(x - 1) + 2(y - 1) - (z - 5) = 0$$

This simplifies to $8x + 2y - z = 5$. For the normal line at the point P_0:

$$x = 1 + 8t \qquad\qquad y = 1 + 2t \qquad\qquad z = 5 - t$$

12. *Many thanks to JD Walsh for writing the question and solution for this problem.*
The direction in which the function increases most rapidly is the direction of the gradient, so we begin by computing the gradient. The partial derivatives are

$$f_x = 2xy + ye^{xy}\sin y$$
$$f_y = x^2 + xe^{xy}\cos y$$

Therefore, the gradient is

$$\nabla f(x,\, y) = (2xy + ye^{xy}\sin y)\mathbf{i} + (x^2 + xe^{xy}\cos y)\mathbf{j}$$

At P_0, the gradient is

$$\nabla f(1,\, 0) = (2(1)(0) + (0)e^{1(0)}\sin 0)\mathbf{i} + ((1)^2 + (1)e^{1(0)}\cos 0)\mathbf{j} = 0\mathbf{i} + 2\mathbf{j} = 2\mathbf{j}$$

The magnitude of the gradient at P_0 is

$$|\nabla f(1,\, 0)| = \sqrt{0^2 + 2^2} = 2$$

Thus, the direction of greatest increase is

$$\frac{\nabla f(1,\, 0)}{|\nabla f(1,\, 0)|} = \mathbf{j}$$

The direction of greatest decrease is the negative of that vector:

$$-\frac{\nabla f(1,\, 0)}{|\nabla f(1,\, 0)|} = -\mathbf{j}$$

The derivative in the direction of greatest increase is the magnitude

$$|\nabla f(1,\, 0)| = 2$$

The derivative in the direction of greatest decrease is the negative of the magnitude

$$-|\nabla f(1,\, 0)| = -2$$

13. *Many thanks to JD Walsh for writing the question and solution for this problem.*

The direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is the unit vector

$$\mathbf{u} = \frac{2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + 2^2 + (-2)^2}} = \frac{2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{\sqrt{12}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$$

The gradient of f is

$$\nabla f = e^x \cos yz\mathbf{i} - ze^x \sin yz\mathbf{j} - ye^x \sin yz\mathbf{k}$$

At the origin, $(0, 0, 0)$, the gradient is

$$\nabla f = \hat{i}$$

Thus the dot product we need is

$$\nabla f|_{P_0} \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$$

Therefore, since $ds = 0.1 = 1/10$,

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})\ ds = \frac{1}{\sqrt{3}}\ \frac{1}{10} = \frac{1}{10\sqrt{3}}$$

X Additional Review Problems for Midterm 3

Sections from Thomas 13^{th} edition: 14.7, 14.8, 14.9, 15.1, 15.2, 15.3, 15.4

Exercises

1. 14.9, Problem 7: Use Taylor's Formula for $f(x,y) = \ln(2x+y+1)$ at the origin to find quadratic and cubic approximation of f near the origin.
2. 15.2 Problem 64: Find the volume of the solid cut from the square column $|x|+|y| \le 1$ by the planes $z=0$ and $3x+z=3$.
3. Evaluate the following integrals.
   (a) $\displaystyle\int_{-7}^{7}\int_{2}^{11} e^{1-x^4} y^7 \cos(c^4+x^4)\,dx\,dy$, where $c \in R$
   (b) $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} \sin\left(\sqrt{x^2+y^2}\right) dy dx$
   (c) 15.4 Problem 48:
   $$\iint_R (x^2+y^2)^{-2} dA$$
   R is the region inside the circle $x^2+y^2=2$ for $x \le -1$.
4. Compute the average value of $f(x,y) = x^5\cos(x^2)$ over the region bounded by $y = 4-|x|$ and $y = x^2$.
5. D is the region enclosed by curves $y = x^2$, $y = 3x$. Construct double integrals in Cartesian coordinates for both orders of integration.
   $$\iint_D xy\,dA$$
6. Construct an integral that represents the volume of the solid below $z = x^2+1$, and above the region inside the circle $r = 3\cos(\theta)$, and outside $r = 1+\cos\theta$.
7. Construct a double integral that represents the area of region bounded by $r^2 = 4\cos 2\theta$.
8. Find the dimensions of the rectangle of greatest area that can be inscribed in the curve $x^2/16+y^2/16 = 1$ with sides parallel to the coordinate axes.
9. Construct iterated integrals to represent the volume of the following solids.
   (a) The tetrahedron in the first octant bounded by the plane $2x+3y+6z = 12$ and the coordinate planes.
   (b) The solid enclosed by the parabolic cylinder $y = 2x^2$ and the planes $z = 0$ and $z = 2-y$.
   (c) The area of the region inside the large loop but outside the smaller loop of $r = 1+2\cos\theta$.
10. Let $D = \{(x,y) \mid x \le y \le 1, 0 \le x \le 1\}$. Evaluate $\displaystyle\iint_D \cos(y^2)dA$.
11. Find all the local maxima, local minima, and saddle points of the function
    $$f(x,\,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$$
12. Use Lagrange Multipliers to identify the point(s) on the surface $z = xy+1$ closest to the origin.

Solutions

Partial solutions to all of the above problems will be posted shortly after the lecture prior to their midterm. Students are encouraged to work on all of the above problems before comparing their work to the solutions.

Partial Solutions

1. 14.9, Problem 7: A screen capture from a solutions manual is below.

$$f(x,y)=\ln(2x+y+1)\Rightarrow f_x=\frac{2}{2x+y+1},\ f_y=\frac{1}{2x+y+1},\ f_{xx}=\frac{-4}{(2x+y+1)^2},\ f_{xy}=\frac{-2}{(2x+y+1)^2},\ f_{yy}=\frac{-1}{(2x+y+1)^2}$$

$$\Rightarrow f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\tfrac{1}{2}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]$$

$$=0+x\cdot 2+y\cdot 1+\tfrac{1}{2}\left[x^2\cdot(-4)+2xy\cdot(-2)+y^2\cdot(-1)\right]=2x+y+\tfrac{1}{2}\left(-4x^2-4xy-y^2\right)$$

$$=(2x+y)-\tfrac{1}{2}(2x+y)^2,\ \text{quadratic approximation;}$$

$$f_{xxx}=\frac{16}{(2x+y+1)^3},\ f_{xxy}=\frac{8}{(2x+y+1)^3},\ f_{xyy}=\frac{4}{(2x+y+1)^3},\ f_{yyy}=\frac{2}{(2x+y+1)^3}$$

$$\Rightarrow f(x,y)\approx \text{quadratic}+\tfrac{1}{6}\left[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\right]$$

$$=(2x+y)-\tfrac{1}{2}(2x+y)^2+\tfrac{1}{6}\left(x^3\cdot 16+3x^2y\cdot 8+3xy^2\cdot 4+y^3\cdot 2\right)$$

$$=(2x+y)-\tfrac{1}{2}(2x+y)^2-\tfrac{1}{3}\left(8x^3+12x^2y+6xy^2+y^2\right)$$

$$=(2x+y)-\tfrac{1}{2}(2x+y)^2+\tfrac{1}{3}(2x+y)^3,\ \text{cubic approximation}$$

2. 15.2 Problem 64: A screen capture from a solutions manual is below.

$$64.\quad V=\int_{-1}^{0}\int_{-x-1}^{x+1}(3-3x)\,dy\,dx+\int_{0}^{1}\int_{x-1}^{1-x}(3-3x)\,dy\,dx=6\int_{-1}^{0}\left(1-x^2\right)dx+6\int_{0}^{1}\left(1-x^2\right)dx=4+2=6$$

3. Solutions to each of the integrals follow.

   (a)

$$\int_{-7}^{7}\int_{2}^{11}e^{1-x^4}y^7\cos(c^4+x^4)dxdy=\int_{-7}^{7}y^7dy\int_{2}^{11}e^{1-x^4}\cos(c^4+x^4)dx$$

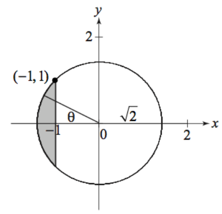
   The first integral is zero because it is the integral of an odd function over an interval symmetric about the origin. The answer is zero.

   (b) We can convert to polar and use integration by parts.

$$\begin{aligned}\int_0^1\int_0^{\sqrt{1-x^2}}\sin\left(\sqrt{x^2+y^2}\right)dydx&=\int_0^{\pi/2}\int_0^1\sin(r)rdrd\theta\\&=\int_0^{\pi/2}(\sin 1-\cos 1)d\theta\\&=(\pi/2)(\sin 1-\cos 1)\end{aligned}$$

   (c) 15.4 Problem 48: A screen capture from a solutions manual is below. The limits for r should be $-\sec\theta$ to $\sqrt{2}$, but the final answer is unaffected by the typo.

48. The region R is shaded in the graph below.



As θ ranges from $3\pi/4$ to $5\pi/4$ the ray at angle θ enters R at $r=\sec\theta$ and leaves R at $r=\sqrt{2}$. Thus

$$\iint_R \left(x^2+y^2\right)^{-2} dA = \int_{3\pi/4}^{5\pi/4}\int_{\sec\theta}^{\sqrt{2}} r^{-4}\cdot r\,dr\,d\theta$$

$$= \int_{3\pi/4}^{5\pi/4} -\frac{1}{2}r^{-2}\Big]_{\sec\theta}^{\sqrt{2}} d\theta = \int_{3\pi/4}^{5\pi/4}\frac{1}{4}\left(2\cos^2\theta-1\right)d\theta = \frac{1}{4}\int_{3\pi/4}^{5\pi/4}\cos 2\theta\,d\theta = \frac{1}{8}\sin 2\theta\Big]_{3\pi/4}^{5\pi/4} = \frac{1}{4}$$

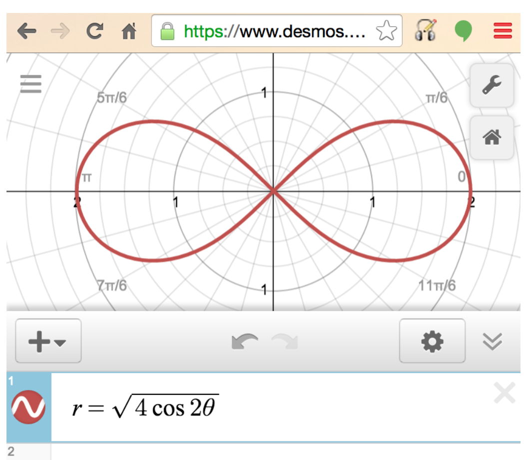
4. The region is symmetric about the y-axis, and the function is odd about the y axis, meaning that $f(x,y) = f(-x,y)$. The average value is zero.

5. The integrals are $\displaystyle\int_0^3\int_{x^2}^{3x} xy\,dy\,dx = \int_0^9\int_{y/3}^{\sqrt{y}} xy\,dx\,dy.$

   If we want to check an answer for such a problem, we can evaluate both and see if they yield the same number.

6. The double integral is $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta}(1+r^2\cos^2\theta)\,r\,dr\,d\theta$

7. Note that the curve $r^2 = 4\cos 2\theta$ is not defined when $\cos 2\theta$ is negative. So for $\theta\in(\pi/4, 3\pi/4)$, the curve is not defined. Also note that $r^2 = 4\cos 2\theta$ implies $r = \pm 2\sqrt{\cos 2\theta}$, so we have both positive and negative values of r for different values of θ. A plot of the curve is shown below.



The area can be found using symmetry: the area of the two loops are equal, and the area of the bottom half is equal to the area of the top half.

$$area = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

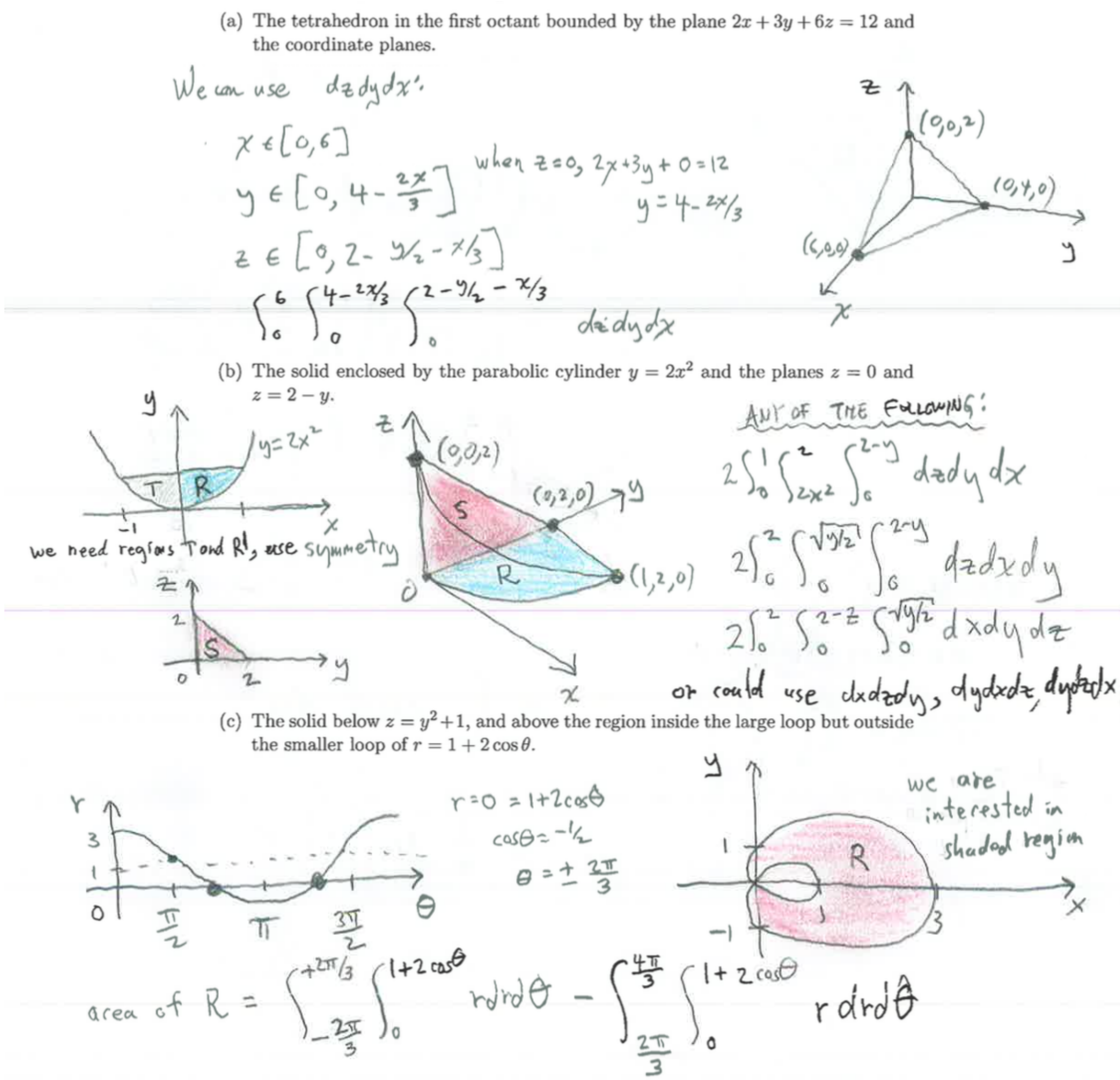
We are not asked to evaluate the integral, but if we were, we would find that the area is equal to 4.

8. The area of the rectangle is $A = 4xy$, and our constraint can be written as $g(x, y) = x^2 + y^2 - 4^2 = 0$. Using Lagrange Multipliers,

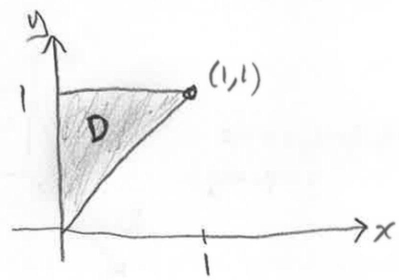
$$\nabla A = \lambda \nabla g, \quad g = x^2 + y^2 - 4^2 = 0$$
$$\begin{bmatrix} 4y \\ 4x \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

The first component gives us $y = \lambda x/2$, which if we substitute into the second component gives us $\lambda = \pm 2$. Therefore $y = \pm x$, but since x and y represent lengths, we only consider $y = x$. Using $x = y$, our constraint becomes $2x^2 = 4^2$, or $x = 2\sqrt{2}$, so the rectangle has width $2x = 4\sqrt{2}$ and width $2y = 4\sqrt{2}$.

9. A screen capture of handwritten solutions is below.



10. A screen capture of handwritten solutions is below.



$$\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$$

$$= \int_0^1 \int_0^y \cos y^2 \, dx\, dy$$

$$= \int_0^1 \cos y^2 \left(\int_0^y 1 \, dx \right) dy$$

$$= \int_0^1 \cos y^2 \, (y - 0)\, dy$$

$$= \int_0^1 y \cos y^2 \, dy$$

use $u = y^2$

$du = 2y\, dy$

$\Rightarrow dy = \dfrac{du}{2y}$

when $y=0$, $u=0$

$y=1$, $u=1$

$$= \int_0^1 \cos u \, \frac{du}{2}$$

$$= \frac{1}{2} \sin(1)$$

11. *Many thanks to JD Walsh for writing the question and solution for this problem.*

We begin by finding the first-order partial derivatives:

$$f_x = 2y - 10x + 4 \qquad\qquad f_y = 2x - 4y + 4$$

Setting both equations equal to zero and combining:

$$\begin{array}{rrrrl} & 2\,(\;-10x & +\,2y & +\;\;4 = 0\;) \\ + & 2x & -\,4y & +\;\;4 = 0 \\ \hline & -18x & & +\,12 = 0 \end{array}$$

So $x = 2/3$. Substituting the x-value into the equation $f_y = 0$ gives

$$2\left(\frac{2}{3}\right) - 4y + 4 = 0$$

This simplifies to $y = 4/3$. Thus, f has one critical point $\left(\frac{2}{3}, \frac{4}{3}\right)$.
To evaluate the behavior of f at the critical point, we need the second-order partial derivatives:

$$f_{xx} = -10 \qquad\qquad f_{yy} = -4 \qquad\qquad f_{xy} = 2$$

This gives us

$$f_{xx}f_{yy} - f_{xy}^2 = -10(-4) - (2)^2 = 40 - 4 = 36 > 0$$

and $f_{xx} = -10 < 0$. By the second derivative test, f has a **local maximum** at the point $\left(\frac{2}{3},\, \frac{4}{3}\right)$.

12. *Many thanks to JD Walsh for writing the question and solution for this problem.*

To find the point closest to the origin, we want to minimize the distance from the origin:

$$f(x,\, y,\, z) = \sqrt{x^2 + y^2 + z^2}$$

Subject to the condition

$$g(x,\, y,\, z) = xy + 1 - z = 0$$

The gradient functions are

$$\begin{aligned} \nabla f(x,\, y,\, z) &= \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{i} + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{j} + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{k} \\ \nabla g(x,\, y,\, z) &= y\mathbf{i} + x\mathbf{j} - \mathbf{k} \end{aligned}$$

Applying the Lagrange Muliplier formula $\nabla f = \lambda \nabla g$:

$$\frac{x}{\sqrt{x^2+y^2+z^2}} = \lambda y \qquad \frac{y}{\sqrt{x^2+y^2+z^2}} = \lambda x \qquad \frac{z}{\sqrt{x^2+y^2+z^2}} = \lambda(-1)$$

Rewriting all three equations gives us

$$\lambda\sqrt{x^2+y^2+z^2} = \frac{x}{y} = \frac{y}{x} = -z$$

From this step on, we can ignore the formula with λ, and consider the other three equations. The description of the problem tells us $z = xy + 1$, so we now have:

$$\frac{x}{y} = \frac{y}{x} = -(xy+1)$$

Because $x/y = y/x$, we know $x^2 = y^2$ and $x = \pm y$. Furthermore, we must have $x/y = y/x = \pm 1$. Thus, the right-hand equation requires that $-(xy+1) = \pm 1$.

If we try $-(xy+1) = -1$, then $xy = 0$, so either x or y is zero. But this is not possible, since we have x and y in the denominators of fractions. Thus, we must take $-(xy+1) = +1$, which tells us $xy = -2$. Therefore, $x = \pm\sqrt{2}$.

If $x = \sqrt{2}$, $y = -\sqrt{2}$, and if $x = -\sqrt{2}$, $y = \sqrt{2}$. In either case, $z = xy + 1 = -1$. Therefore, we have two potential points:

$$(-\sqrt{2},\, \sqrt{2},\, -1) \qquad \text{and} \qquad (\sqrt{2},\, -\sqrt{2},\, -1)$$

The distances are

$$\begin{aligned} \mathrm{dist}(-\sqrt{2},\, \sqrt{2},\, -1) &= \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2 + (-1)^2} = \sqrt{5} \\ \mathrm{dist}(\sqrt{2},\, -\sqrt{2},\, -1) &= \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2 + (-1)^2} = \sqrt{5} \end{aligned}$$

Therefore, both points are equally close to the origin, and the solution is

$$(-\sqrt{2},\, \sqrt{2},\, -1) \qquad \text{and} \qquad (\sqrt{2},\, -\sqrt{2},\, -1)$$