Math 2552, Exam 2 (Practice) Name and section: ____________________

1. (4.3.5)

Consider the second order linear equation for a function $y = y(x)$:

$$y'' - 2y' + 2y = 0.$$

(a) (5 points) Find an **equivalent first order linear system**.

(b) (5 points) Write the system in matrix form.

(c) (10 points) Find the **general solution** of the system.

Solution:

(a) $y' = z$, $z' = -2y + 2z$.

(b)

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

(c) The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$ with roots $1 \pm i$. Taking the eigenvalue $1 - i$, and

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix},$$

we find

$$A - (1-i)I = \begin{pmatrix} -1+i & 1 \\ -2 & 1+i \end{pmatrix}.$$

It is easy to check that the second row is a (complex) multiple of the first, and a corresponding complex eigenvector is

$$\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

This tells me that the inverse of the change of basis matrix is

$$N^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

and the solution is

$$\begin{aligned} \mathbf{x} &= e^t N^{-1} \left[c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right] \\ &= e^t \left[c_1 \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} - c_2 \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix} \right]. \end{aligned}$$

Notice that I didn't require in this question, nor include in the solution, a phase plane diagram, but it would be worthwhile to make sure you could provide one. Note also that in this case, the transformation corresponding to the change of basis is orientation reversing, so the rotation is in the opposite direction from that of the standard (canonical form) system.

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2. (20 points) (8.3.3)

Find the **improved Euler approximation** for $y(0.2)$ with stepsize $h = 0.1$ if y solves the IVP

$$\begin{cases} y' = 2y - 3t \\ y(0) = 1. \end{cases}$$

Solution:

1. First improved Euler step starting from $(t_0, Y_0) = (0, 1)$:

   (a) Euler approximation at $t_1 = t_0 + h = 0.1$ starting with $Y_0 = y_0 = 1$:

   $$m_0 = 2, \qquad y_1 = 1 + 2/10 = 6/5.$$

   (b) A second slope:

   $$m_1 = 2y_1 + 3t_1 = 2(6/5) - 3/10 = 21/10.$$

   (c) The average of these two slopes:

   $$m = (m_0 + m_1)/2 = (2 + 21/10)/2 = 41/20.$$

   (d) (first) improved Euler step:

   $$Y_1 = Y_0 + mh = 1 + 41/200 = 241/200.$$

2. Second improved Euler step starting from $(t_1, Y_1) = (0.1, 241/200)$:

   (a) Euler approximation at $t_2 = t_1 + h = 0.2$ starting with $y_0 = Y_1 = 241/200$:

   $$\begin{aligned} m_0 &= 2(241/200) - 3/10 = 211/100, \\ y_1 &= 241/200 + 211/1000 = 2832/2000 = \frac{1416}{10^3} = 1.416. \end{aligned}$$

   (b) A second slope:

   $$m_1 = 2y_1 + 3(0.2) = \frac{1416}{(2^2)(5^3)} - \frac{6}{10} = \frac{354}{5^3} - \frac{3}{5} = \frac{279}{5^3}.$$

   (c) The average of these two slopes:

   $$m = (m_0 + m_1)/2 = \frac{1}{2}\left[\frac{211}{(2^2)(5^2)} + \frac{279}{5^3}\right] = \frac{1}{2}\left[\frac{1055 + 1116}{(2^2)(5^3)} = \frac{2171}{10^3}\right].$$

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(d) (second) improved Euler step:

$$
\begin{aligned}
Y_2 = Y_1 + mh &= \frac{241}{2(10^2)} + \frac{2171}{10^4} \\
&= \frac{241}{(2^3)(5^2)} + \frac{2171}{(2^4)(5^4)} \\
&= \frac{12050 + 2171}{(2^4)(5^4)} \\
&= \frac{14221}{(2^4)(5^4)} = 1.4221.
\end{aligned}
$$

This is the (final) answer.

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3. (Section 7.2: simple pendulum with nonlinear cubic damping) A nonlinear damped pendulum is modeled by
$$\theta'' = -2\sin\theta - 2\tan^{-1}\theta'.$$

   (a) (5 points) Find an **equivalent first order nonlinear system**.

   (b) (5 points) Find all equilibrium points of your system and the linearization at each equilibrium point.

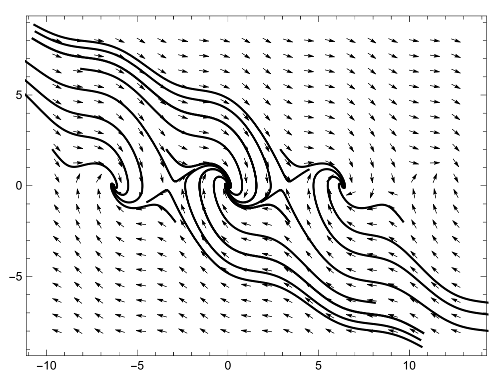
   (c) (10 points) Draw the global phase diagram for your system.

Solution:

(a)
$$\begin{cases} \theta' = z \\ z' = -2\sin\theta - 2\tan^{-1} z. \end{cases}$$

(b) All equilibrium points $(\theta_*, z_*)^T$ have $z_* = 0$. Therefore, there is an infinite sequence of equilibrium points $(\theta_*, z_*)^T = (\pi k, 0)^T$ for $k \in \mathbb{Z}$ just as in the case of the linearly damped (or undamped) simple pendulum. The derivative of the vector field in this case is
$$D\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -2\cos\theta & -2/(1+z^2) \end{pmatrix}.$$
Therefore, for even multiples $\theta_* = 2\pi k$, the linearized system is
$$\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\mathbf{x} \qquad \text{(this represents a clockwise spiral sink).}$$
For odd multiples $\theta_* = (2k+1)\pi$, the linearized system is
$$\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}\mathbf{x} \qquad \text{(this represents an unstable saddle as usual).}$$

(c)



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4. (20 points) (7.4.9) Draw the phase diagram within the closed first quadrant $Q = \{(x, y) : x \geq 0,\ y \geq 0\}$ for this nonlinear autonomous system:

$$\begin{cases} x' = x(1 - y/2) \\ y' = y(-1 + x/3). \end{cases}$$

Solution:

1. The equilibrium points are at $(0,0)^T$ and $(3,2)^T$.
2. The derivative of the field is
$$D\mathbf{F} = \begin{pmatrix} 1 - y/2 & -x/2 \\ y/3 & -1 + x/3 \end{pmatrix}.$$
3. At the origin, the linearization is
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \qquad \text{(a saddle).}$$
Observe also that the axes are both orbits. Restriction to the x-axis gives $x' = x$ (expansion). Restriction to the y-axis gives $y' = -y$ (contraction).
4. The linearization at the other equilibrium point is
$$\mathbf{x}' = \begin{pmatrix} 0 & -3/2 \\ 2/3 & 0 \end{pmatrix} \mathbf{x} \qquad \text{(a periodic center).}$$
5. Given the simplicity of the system, periodic orbits are expected.
Formal calculation:
$$\frac{dy}{dx} = \frac{y(-1 + x/3)}{x(1 - y/2)}.$$
$$\left(\frac{1}{y} - \frac{1}{2}\right)\frac{dy}{dx} = -\frac{1}{x} + \frac{1}{3}.$$
From this, we guess that $H(x, y) = \ln y - y/2 + \ln x - x/3$ is conserved (along orbits).
Confirmation:
$$\frac{d}{dt}H(x,y) = (1/y - 1/2)y' + (1/x - 1/3)x' = (1 - y/2)(-1 + x/3) + (1 - x/3)(1 - y/2) = 0.$$

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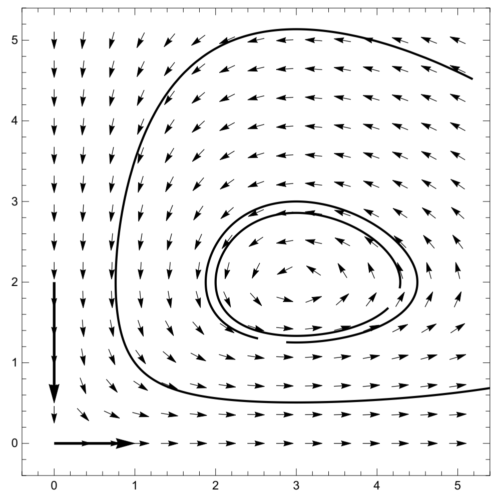
6. The gradient of H satisfies $\nabla H = (1/x - 1/3, 1/y - 1/2)$, and the Hessian is

$$D^2 H = \begin{pmatrix} -1/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}.$$

From this, we see that H has a unique maximum at the first quadrant equilibrium point, and we have periodic orbits.



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5. (20 points) (4.5.8) Find the general solution of

$$y'' - y' - 2y = 2e^{-t}.$$

Solution: We start by finding the general homogeneous solution using $y_h = e^{\alpha t}$. This leads to

$$y_h = c_1 e^{-t} + c_2 e^{2t}.$$

A first guess for a particular solution is $y_p = ce^{-t}$, but this is in the kernel of the operator, so we use $y_p = cte^{-t}$ instead. Plugging this in, we find

$$c[-2 + t - 1 + t - 2t] = 2 \qquad \text{or} \qquad c = -2/3.$$

The general solution is

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^{2t} - \frac{2t}{3}\, e^{-t}.$$