Lecture 10 (Fsections)

Two Main Things:

1. Inhomogeneous systems 3.2.15-20
2. stability (2D)

3.2.16

$$\begin{cases} x' = -x - 4y - 4 \\ y' = x - y - 6 \end{cases} \quad \Bigg| \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

General Case: $\mathbb{X}' = A\mathbb{X} + \mathbb{b}$

constant

$$\mathbb{x}' = \underline{\underline{A}}\mathbb{x} + \mathbb{b}$$

equilibrium $A\mathbb{x}^* = -\mathbb{b}$

maybe just one (nonzero) solution if A is invertible

$$\tilde{\mathbb{x}}(t) = \mathbb{x}(t) - \mathbb{x}^*$$

solution

constant

$$\tilde{\mathbb{x}}' = \mathbb{x}' = A\mathbb{x} + b = A(\mathbb{x} - \mathbb{x}_*) = A\tilde{\mathbb{x}}$$

$\tilde{\mathbb{x}}' = A\tilde{\mathbb{x}}$ ← Associated homogeneous ODE(s)

Given $\mathbb{X}' = \underset{\text{constant}}{A}\mathbb{X} + \mathbb{b}$ and $\mathbb{X}^* = -A^{-1}\mathbb{b}$

The <u>general solution</u> is $\mathbb{X} = \tilde{\mathbb{X}} + \mathbb{X}^*$

where $\tilde{\mathbb{X}}$ is the general solution of the assoc. homogeneous eqn.

$$\tilde{\mathbb{X}}' = A\tilde{\mathbb{X}}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

<u>equilibrium</u> $\begin{pmatrix} x_* \\ y_* \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 20 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Assoc. Hom. System $\cancel{\lambda^2+2\lambda}$ $(\lambda+1)^2=-4$

later $\longrightarrow$ Complex eigenvalues.

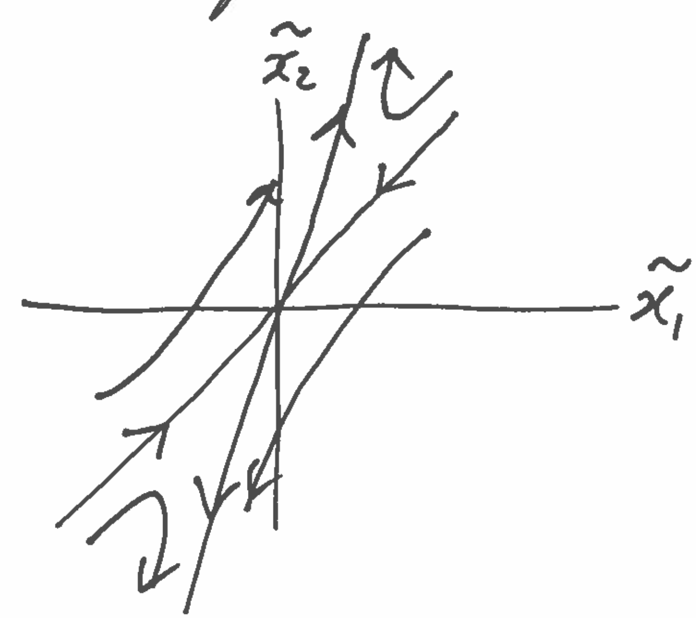
3.2.18 $\begin{cases} x' = -2x+y-11 \\ y' = -5x+4y-35 \end{cases}$ $A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$

$\lambda^2-2\lambda-3=0$, $(\lambda+1)(\lambda-3)=0$, $\lambda = -1, 3$

Saddle for associated homogeneous system

$\lambda=-1$ $\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix}$ $V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda=3$ $\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix}$ $W = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$



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$$\tilde{X} = c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\5\end{pmatrix}$$

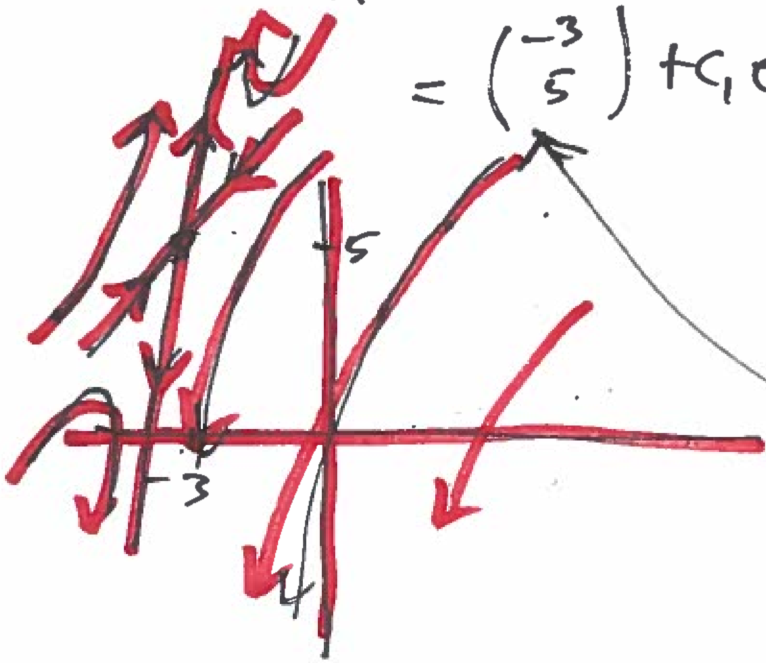
$$\begin{cases}\tilde{x}_1 = c_1 e^{-t} + c_2 e^{3t}\\ \tilde{x}_2 = c_1 e^{-t} + 5 c e^{3t}\end{cases}$$ Solution of $\tilde{X}' = \begin{pmatrix}-2 & 1\\-5 & 4\end{pmatrix}\tilde{X}$

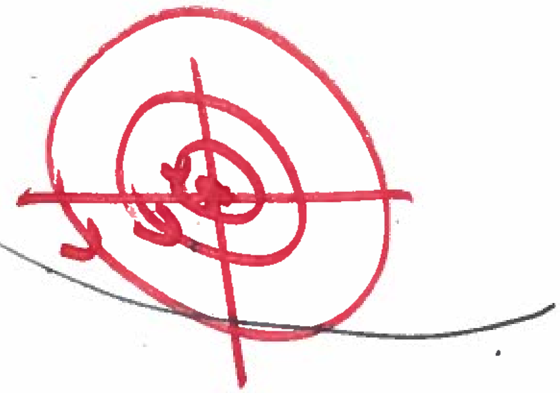
General Solution of $X' = \begin{pmatrix}-2 & 1\\-5 & 4\end{pmatrix}X - \begin{pmatrix}11\\35\end{pmatrix}$

$$X = X^* + c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\5\end{pmatrix}$$ where

$$= \begin{pmatrix}-3\\5\end{pmatrix} + c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\5\end{pmatrix}$$

$$X^* = -\frac{1}{3}\begin{pmatrix}4 & -1\\5 & -2\end{pmatrix}\begin{pmatrix}11\\35\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}9\\-15\end{pmatrix} = \begin{pmatrix}-3\\5\end{pmatrix}$$

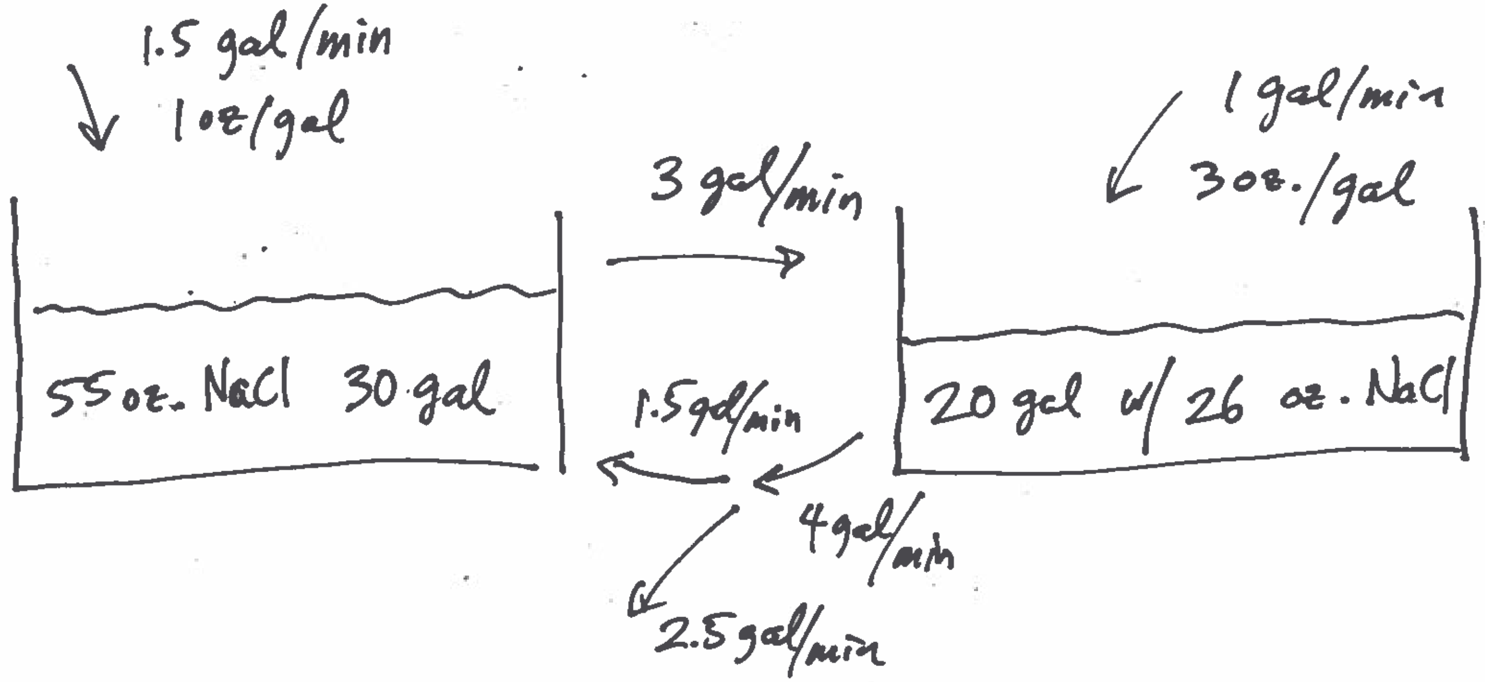




$$A^{-1} = \frac{1}{\det A}\left(A^{cof}\right)^T$$

$$A^{cof} = \begin{array}{cc} & j \\ i & \begin{pmatrix} + & - & \cdots & & \\ - & + & & & \\ \vdots & & & & \\ & & \sigma\, \boxed{\det} \check{A}_{ij} & & \\ & & & & \end{pmatrix} \end{array}$$

3.2.30



$$\begin{cases} S_1' = 1.5 - \dfrac{3 S_1}{30} + \dfrac{1.5 S_2}{20}, & S_1(0) = 55 \\[2ex] S_2' = 3 + 3\dfrac{S_1}{30} - 4\dfrac{S_2}{20}, & S_2(0) = 26 \end{cases}$$

$$A = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} = \frac{1}{40}\begin{pmatrix} -4 & 3 \\ 4 & -8 \end{pmatrix}.$$

equilibrium

$$\frac{1}{40}\begin{pmatrix} -4 & 3 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} = \begin{pmatrix} -3/2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} = 60 \, \frac{1}{20}\begin{pmatrix} -8 & -3 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= 3\begin{pmatrix} 8 & 3 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 14 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} 42 \\ 36 \end{pmatrix}$$

Conjecture

$$\frac{42}{30} \stackrel{?}{=} \frac{36}{20}$$

$$\frac{7}{5}, \quad \frac{9}{5}$$

Stabilize at nonequal concentrations?

Associated Homogeneous System

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix}$$

$$\| \qquad\qquad \|$$

$$\cancel{\tilde{x}}' \qquad\qquad \cancel{\tilde{x}}$$

$$\left(-\frac{1}{10}-\lambda\right)\left(-\frac{1}{5}-\lambda\right) - \frac{3}{4(10)^2} = 0$$

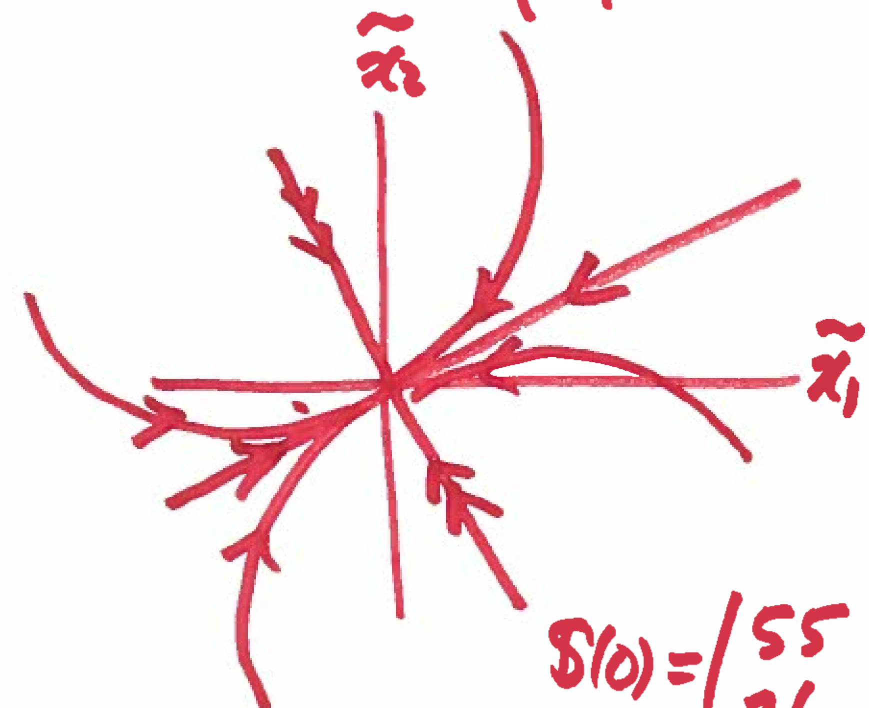
$$\lambda^2 + \frac{3}{10}\lambda + \underbrace{\frac{1}{5\cdot 10}}_{=\frac{8}{4\cdot 10^2}} - \frac{3}{4\cdot 10^2} = \lambda^2 + \frac{3}{10}\lambda + \underbrace{\frac{5}{4\cdot 10^2}}_{=\frac{1}{80}} = 0$$

$$\boxed{80\lambda^2 + 24\lambda + 1 = 0} \qquad (20\lambda+1)(4\lambda+1) = 0$$

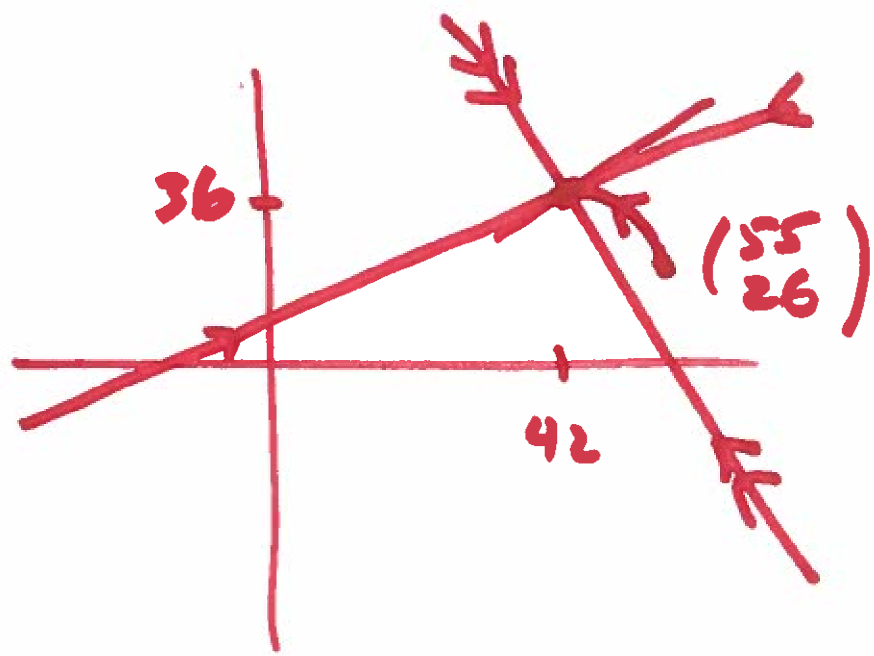
$\underline{\lambda = -\frac{1}{4}}$ $\begin{pmatrix} 3/20 & 3/40 \\ 1/10 & 1/20 \end{pmatrix}$ $2v_1 = -v_2$

$V = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{\lambda = -\frac{1}{20}}$ $\begin{pmatrix} -\frac{1}{20} & \frac{3}{40} \\ \frac{1}{10} & -\frac{3}{20} \end{pmatrix}$ $W = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

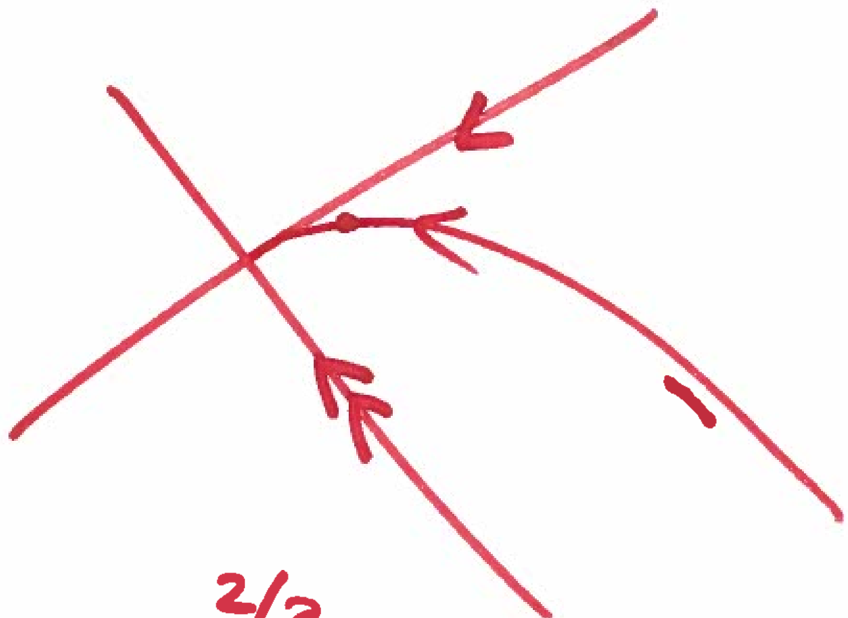


$S(0) = \begin{pmatrix} 55 \\ 26 \end{pmatrix}$



$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + c_1 e^{-\frac{t}{4}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-\frac{t}{20}} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} x' = -3x \\ y' = -2y \end{cases}$$



$y = x^{2/3}$

$x = ce^{-3t}$

$y = ce^{-2t}$