Lecture 10 (2 sections)

1. Inhomogeneous systems (3.2.15-20)
2. stability (2D)

$$\begin{cases} x' = -2x + y - 11 \\ y' = -5x + 4y - 35 \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 11 \\ 35 \end{pmatrix}$$

General Case

const.

$$\mathbb{X}' = A\mathbb{X} + \mathbb{b}$$

$$L\mathbb{X} = \mathbb{X}' - A\mathbb{X}$$

constant inhomogeneity $\mathbb{b} \neq 0$.

EQUILIBRIUM:

$$A\mathbb{X}^* = -\mathbb{b}$$

unique $\mathbb{X}^*$ if A is invertible ($\det A \neq 0$).

$$\tilde{\mathbb{X}} = \mathbb{X} - \mathbb{X}^*$$

soln (unknown)

const.

$$\tilde{\mathbb{X}}' = \mathbb{X}' = A\mathbb{X} + \mathbb{b} = A(\mathbb{X} - \mathbb{X}_*) = A\tilde{\mathbb{X}}$$

$$\tilde{\mathbb{X}}' = A\tilde{\mathbb{X}}$$

Given $\mathbb{X}' = A\mathbb{X} + \mathbb{b}$ (to solve)

Find $\mathbb{X}^* = -A^{-1}\mathbb{b}$.

The general solution is $\mathbb{X} = \tilde{\mathbb{X}} + \mathbb{X}_*$

where $\mathbb{X}^*$ is the general solution of $\boxed{\tilde{\mathbb{X}}' = A\tilde{\mathbb{X}}}$

ASSOCIATED HOMOGENEOUS SYSTEM

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 11 \\ 35 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 11 \\ 35 \end{pmatrix} \longrightarrow \begin{pmatrix} x^* \\ y^* \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 11 \\ 35 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 9 \\ -15 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 5 \end{pmatrix} \text{ (equilibrium).}$$

$$A^{-1} = \frac{1}{\det A}\left(A^{cof}\right)^T$$

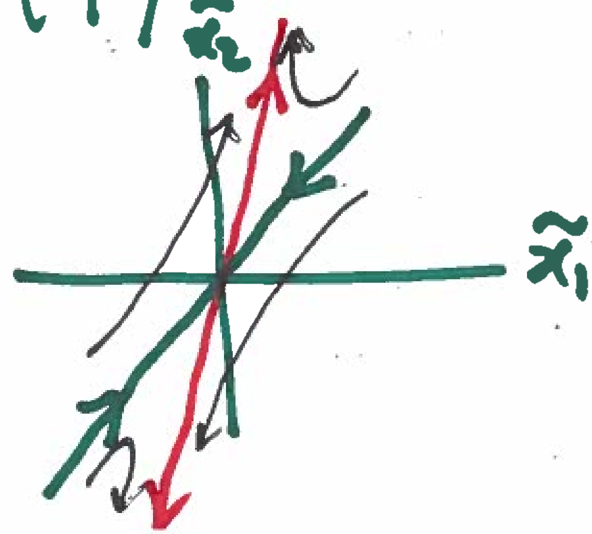
$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda+1)(\lambda-3) = 0$$

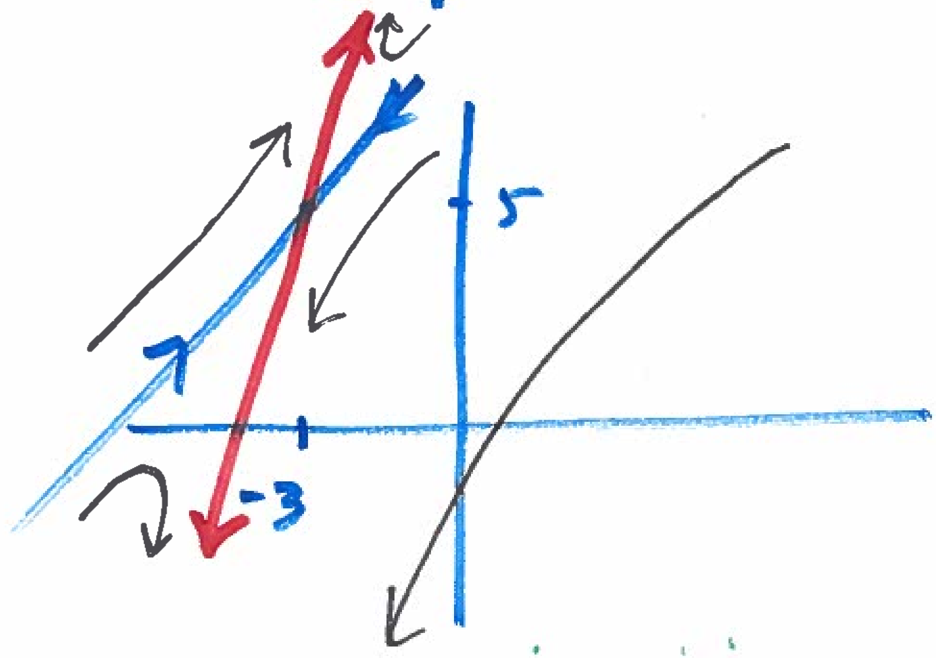
$$\underline{\underline{-1}} \quad \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \qquad \mathbb{V} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\underline{\underline{3}} \quad \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \qquad \mathbb{W} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

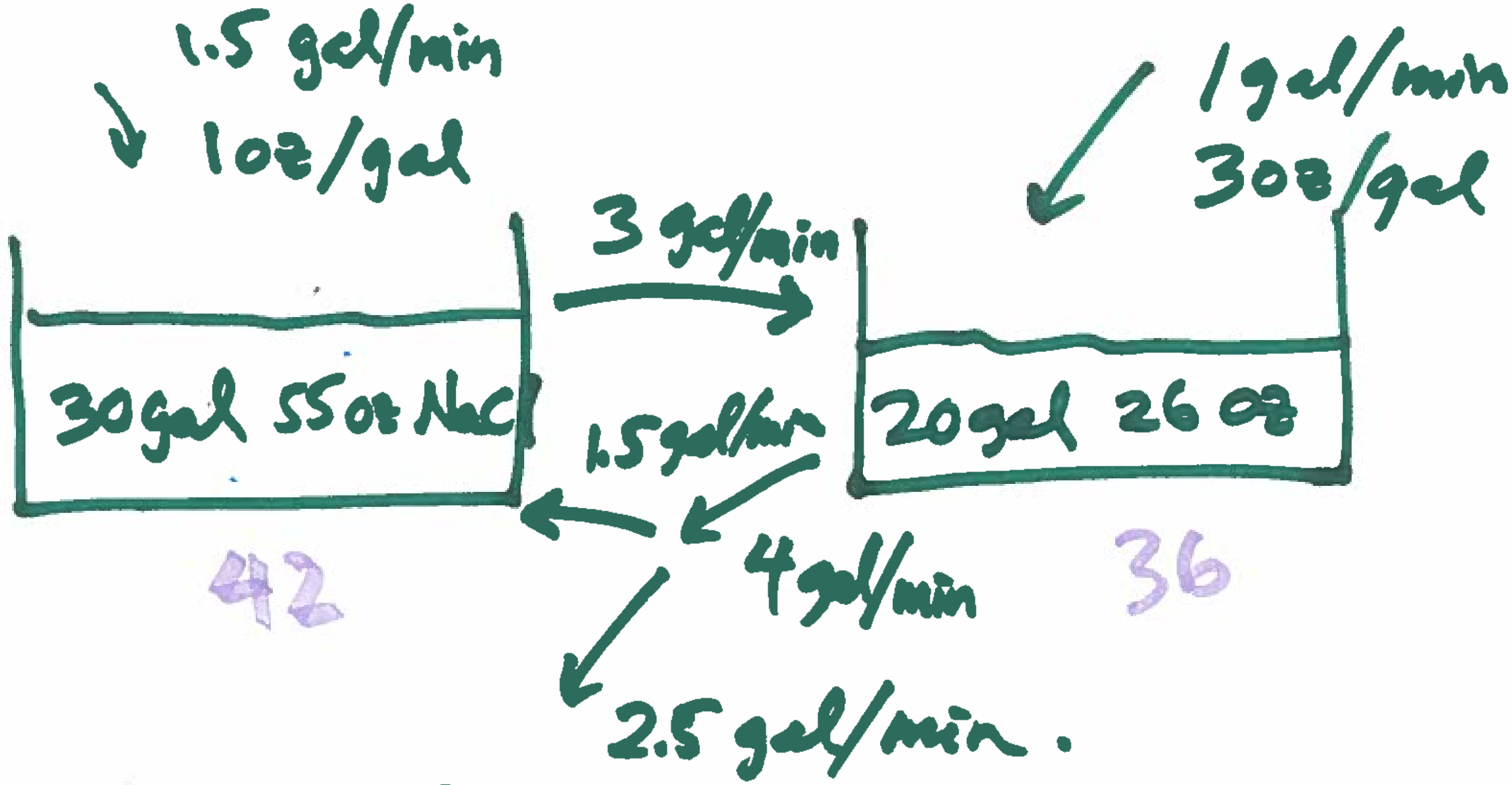
$$\tilde{X} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



$$\mathbb{X} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



3.2.30



$$\begin{cases} S_1' = 1.5 - 3\dfrac{S_1}{30} + 1.5\dfrac{S_2}{20}, & S_1(0) = 55 \\ S_2' = 3 + 3\dfrac{S_1}{30} - 4\dfrac{S_2}{20}, & S_2(0) = 26 \end{cases}$$

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}' = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

EQUILIBRIUM

$$\frac{1}{40}\begin{pmatrix} -4 & 3 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} = \begin{pmatrix} -3/2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 3 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} = -60\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} = \frac{1}{20}\begin{pmatrix} +8 & 3 \\ 4 & 4 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cdot 60$$

$$= 3\begin{pmatrix} 14 \\ 12 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 42 \\ 36 \end{pmatrix}}}$$

Associated Homogeneous System

$$\tilde{\mathbb{X}}' = \begin{pmatrix} -1/10 & 3/40 \\ 1/10 & -1/5 \end{pmatrix} \tilde{\mathbb{X}}$$

$$\left(-\frac{1}{10}-\lambda\right)\left(-\frac{1}{5}-\lambda\right) - \frac{3}{4(10)^2} = 0$$

$$\lambda^2 + \frac{3}{10}\lambda + \frac{1}{50} - \frac{3}{4(10)^2} = 0$$

$$\frac{1}{50} = \frac{8}{4(10)^2}$$

$$\lambda^2 + \frac{3}{10}\lambda + \frac{5}{4(10)^2} = 0 \rightarrow \boxed{80\lambda^2 + 24\lambda + 1 = 0}$$

$$\frac{5}{4(10)^2} = \frac{1}{80}$$

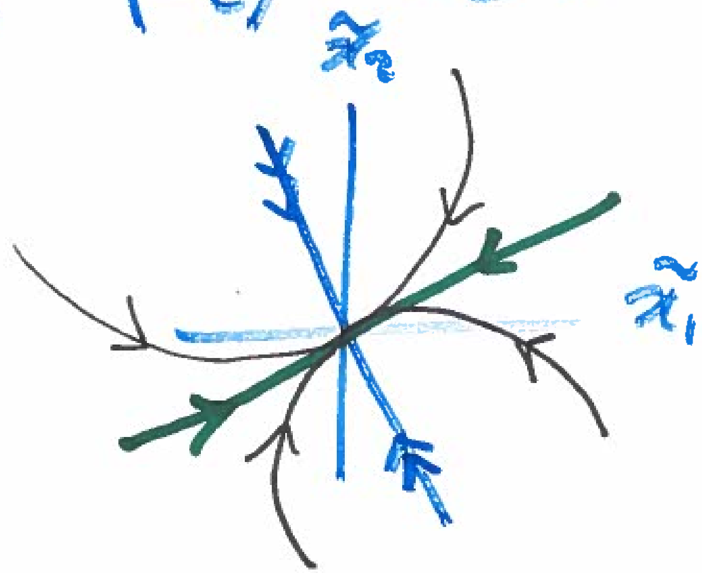
$$(4\lambda+1)(20\lambda+1) = 0.$$

$\underline{\lambda = -\frac{1}{4}}$ $\quad \begin{pmatrix} 3/20 & 3/40 \\ 1/10 & 1/20 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = -\frac{1}{20} \quad \begin{pmatrix} -1/20 & 3/40 \\ 1/10 & -3/20 \end{pmatrix} \quad w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$2v_1 = 3v_2$

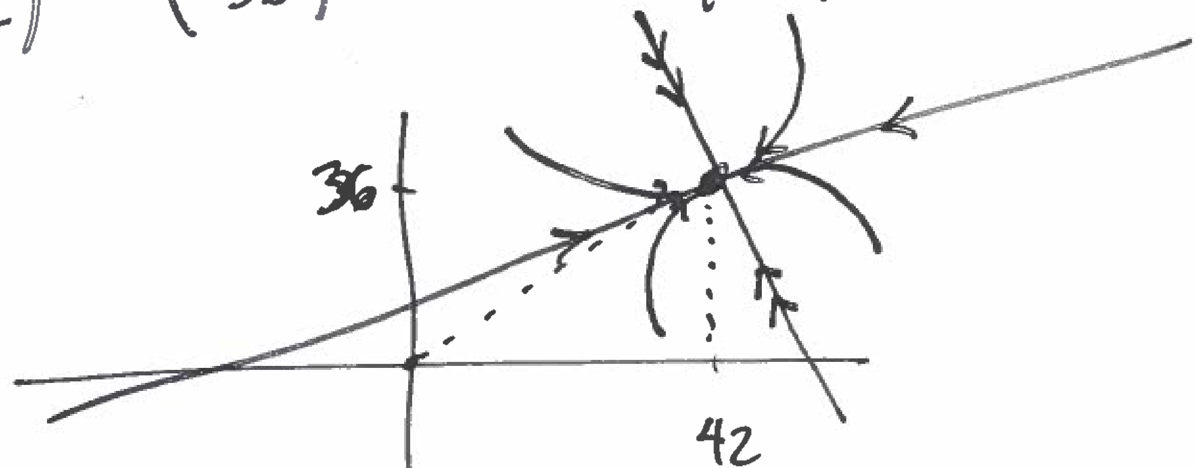
$\tilde{x} = c_1 e^{-t/4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t/20}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$



Original System

IVP $\begin{cases} S_1' = 1.5 - 3\dfrac{S_1}{30} + 1.5\dfrac{S_2}{20}, & S_1(0) = 55 \\ S_2' = 3 + 3\dfrac{S_1}{30} - 4\dfrac{S_2}{20}, & S_2(0) = 26 \end{cases}$

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + c_1 e^{-t/4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t/20}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



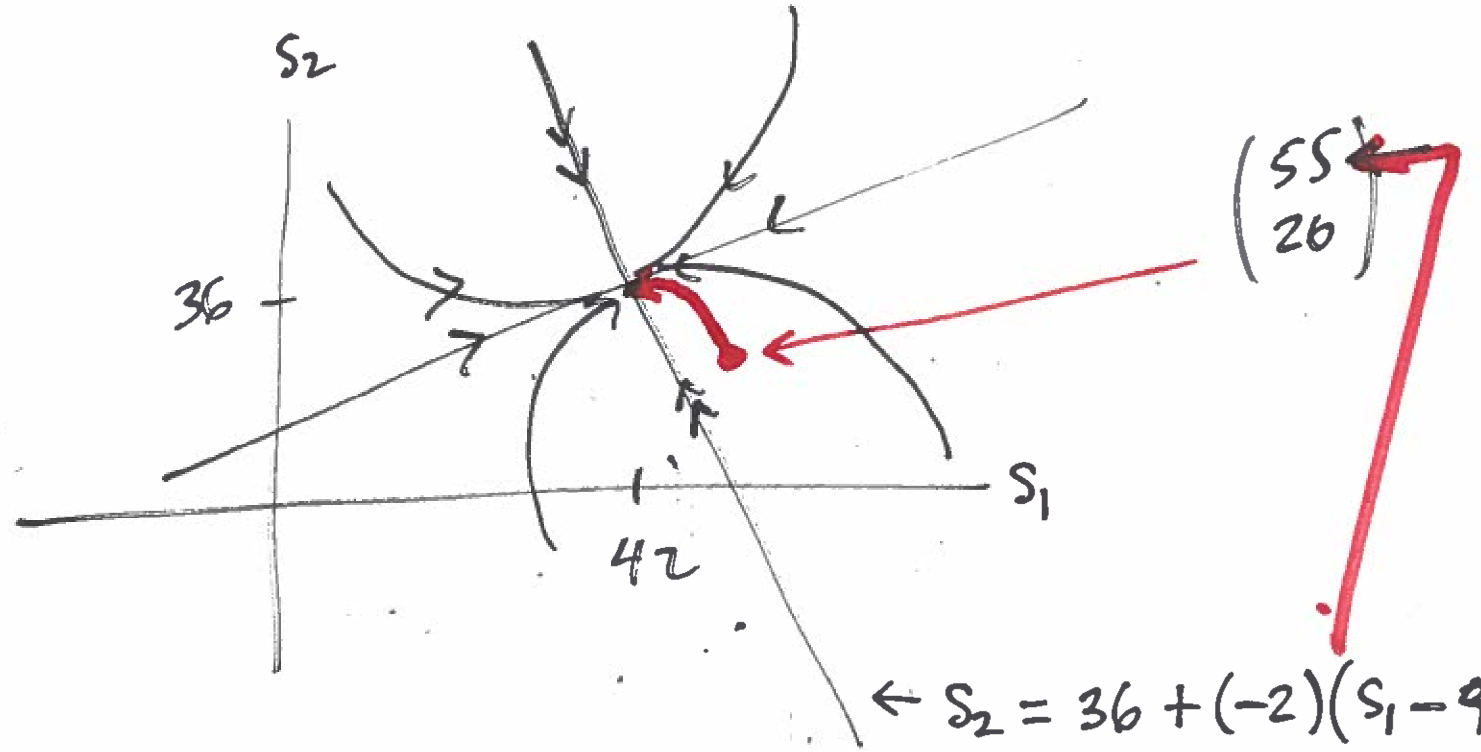
$$\frac{36}{42} = \frac{6}{7}$$

$$\frac{2}{3}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 13 \\ -10 \end{pmatrix}; \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ -10 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 56 \\ 16 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + 7e^{-t/4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2e^{-t/20}\begin{pmatrix} 3 \\ 2 \end{pmatrix}} \leftarrow \text{IVP}$$



$\leftarrow s_2 = 36 + (-2)(s_1 - 42)$

$36 - 2(13) = 10$

1.1.17

\$20,000 loan at 5% annual interest (compounded continuously) \$200/month payment.

Amount Owed:

$$\begin{cases} A' = -2400 + \frac{5}{100}A \\ A(0) = 20{,}000 \end{cases}$$

$$\left(e^{-\frac{5t}{100}}A\right)' = -2400e^{-\frac{5t}{100}}$$

$$e^{-\frac{5t}{100}}A = 20{,}000 + \frac{24(10)^4}{5}e^{-\frac{5t}{100}} - \frac{24(10)^4}{5}$$

$$A = \left[2(10)^4 - \frac{24(10)^4}{5}\right]e^{\frac{5t}{100}} + \frac{24(10)^4}{5}$$

$$= \frac{24}{5}(10)^4 - \frac{14(10)^4}{5}e^{5t/100}$$

$$12 - 7e^{\frac{5T}{100}} = 0$$

$$T = \frac{100}{5}\ln\left(\frac{12}{7}\right) \approx 10.78 \text{ yrs.}$$

$$A \xrightarrow{1 \text{ yr.}} (1+r)A$$

$$A \xrightarrow{6 \text{ mos.}} \left(1+\frac{r}{2}\right)A \longrightarrow \left(1+\frac{r}{2}\right)A + \frac{r}{2}\left(1+\frac{r}{2}\right)A$$

$$\left(1+\frac{r}{2}\right)^2 A$$

4 mos.

k times

$$\left(1+\frac{r}{k}\right)^k A$$

$$\big\downarrow {\scriptstyle k \nearrow \infty}$$

$$e^r A$$

$$Ae^{rt}$$