# Lecture 8

$$y' = \alpha y\left(1 - \frac{y}{K}\right)$$

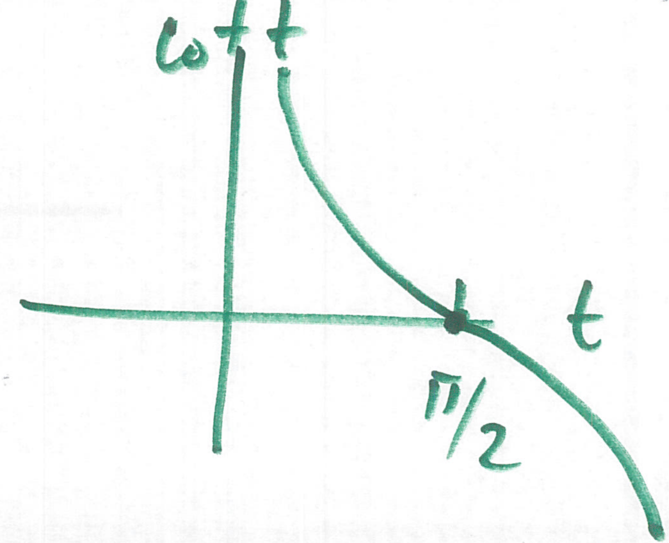

phase line.

# Existence/Uniqueness

__Example__ $t\,y' = \cot t + y$

__Coefficients__: $y' = \frac{1}{t}\,y + \frac{1}{t}\cot t$

If $\underline{y(\pi/2) = 6}$, then the solution is guaranteed to exist (and be unique) on $\underline{\underline{(0, \pi)}}$.



# Linear existence/uniqueness (for systems)

$$L\mathbb{X} = \mathbb{X}' - A\mathbb{X}, \quad \mathbb{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \quad L\mathbb{X} = \mathbb{f}(t) = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

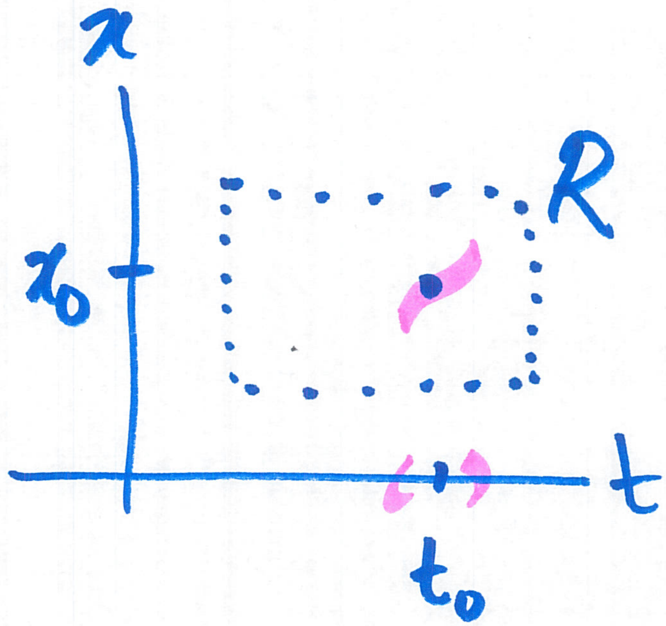
**Theorem** If $a_{ij} = a_{ij}(t)$ and $f_j = f_j(t)$ are <u>continuous</u> on $(a,b) \subseteq \mathbb{R}$, and $t_0 \in (a,b)$, then the IVP

$$\begin{cases} \mathbb{X}' = A\mathbb{X} + \mathbb{f} \\ \mathbb{X}(t_0) = \mathbb{X}_0 \end{cases}$$

has a unique solution defined for $t \in (a,b)$ and (any) $\mathbb{X}_0 \in \mathbb{R}^n$.

# Nonlinear existence/uniqueness (single ODE)

**Theorem** If $R$ is an open region in $\mathbb{R}^2$ and $(t_0, x_0) \in R$ <u>and</u> $f$ is continuous on $R$ <u>and</u> $\frac{\partial f}{\partial x}$ is continuous on $R$, <u>then</u> there is some $r > 0$ for which the IVP




$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution defined for $t_0 - r < t < t_0 + r$.

Nonlinear existence/uniqueness (systems)

**Theorem** If $R$ is an open region in $\mathbb{R}^{n+1}$ and $(t_0, \mathbb{X}_0) \in R$ and $\mathbb{F}$ is continuous on $R$

($f_1, \dots, f_n$ are cont. on $R$)

and $\dfrac{\partial f_i}{\partial x_j}$ is continuous on $R$ for $i, j = 1, \dots, n$,

**THEN** there is some $r > 0$ for which the IVP

$$\begin{cases} \mathbb{X}' = \mathbb{F}(t, \mathbb{X}) \\ \mathbb{X}(\cancel{t}\, t_0) = \mathbb{X}_0 \end{cases}$$

has a unique solution defined for $t_0 - r < t < t_0 + r$.

## 3.3.1

$$\mathbb{X}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbb{X}$$

$\uparrow$ Linear constant coefficient system (homogeneous)

$$\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Important FACTOID

If $(\lambda, \mathbb{V})$ is an eigenvalue/eigenvector pair for $A$, then $\boxed{\mathbb{X}(t) = c e^{\lambda t} \mathbb{V}}$ is a (straight line) solution for $\mathbb{X}' = A\mathbb{X}$.

(Works for $A$ = const. matrix.)

Why?

$\mathbb{X}(t) = \underset{\uparrow \text{ constant}}{c}\, e^{\lambda t} \mathbb{V}$; $\quad \mathbb{X}' = c \lambda e^{\lambda t} \mathbb{V}$

$= c e^{\lambda t} A \mathbb{V}$ (matrix mult is linear)

$= A(c e^{\lambda t} \mathbb{V}) = A\mathbb{X}$

<u>More generally</u>, If $A$ is a constant matrix and $(\lambda_1, \mathbb{V}_1), (\lambda_2, \mathbb{V}_2), \ldots, (\lambda_n, \mathbb{V}_n)$ is a collection of eigenvalue/eigenvector pairs for $A$ <u>with</u>

$$\{\mathbb{V}_1, \ldots, \mathbb{V}_n\} \text{ a } \underline{\text{basis}} \text{ for } \mathbb{R}^n,$$

Then the <u>general solution</u> of $\mathbb{X}' = A\mathbb{X}$ is

$$\mathbb{X}(t) = c_1 e^{\lambda_1 t}\mathbb{V}_1 + c_2 e^{\lambda_2 t}\mathbb{V}_2 + \cdots + c_n e^{\lambda_n t}\mathbb{V}_n$$

$$= \sum_{j=1}^{n} c_j \underbrace{e^{\lambda_j t}\mathbb{V}_j}$$

**Example 3.3.1** $\mathbf{X}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{X}$

$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = \lambda^2 - \lambda - 2 \rightarrow \lambda = \frac{1 \pm \sqrt{1+8}}{2}$

$\hookrightarrow (\lambda - 2)(\lambda + 1) \rightarrow -1, 2$

**Eigenvectors:**

$\lambda = -1$ $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \leftarrow$ eigenvectors are in the kernel/nullspace of this matrix

$4v_1 - 2v_2 = 0 \longrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\left(-1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ is an eigenvalue/eigenvector pair

$\mathbb{X}(t) = c e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$ is a solution.

<u>check</u> $\begin{cases} x_1(t) = c e^{-t} \\ x_2(t) = 2c e^{-t} \end{cases}$

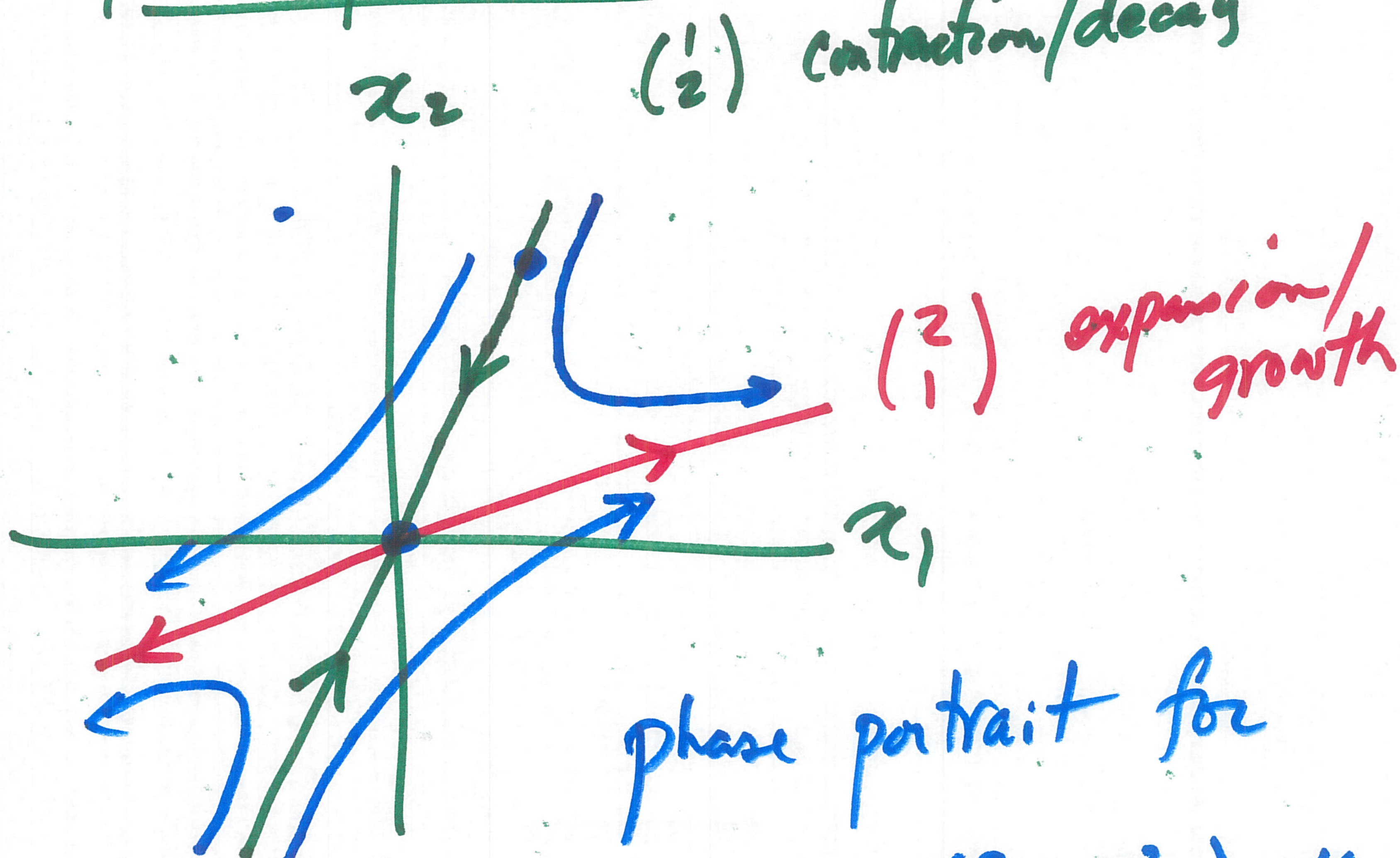
$\begin{cases} x_1' = -ce^{-t}, & 3x_1 - 2x_2 = -ce^{-t} \checkmark \\ x_2' = -2ce^{-t}, & 2x_1 - 2x_2 = -2ce^{-t} \checkmark \end{cases}$

$\underline{\underline{\lambda_2 = 2}}$ $\begin{pmatrix}1 & -2\\2 & -4\end{pmatrix}$ eigenvector $\begin{pmatrix}2\\1\end{pmatrix}$

$$\mathbb{X}(t) = c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$$

$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

# phase portrait




phase portrait for

$$X' = \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix} X .$$