Lecture 8

Review: Nonlinear Existence & Uniqueness

$(t_0, \mathbf{x}_0) \in R$

If R is an open region in $\mathbb{R}^{n+1}$ and $\cancel{(t_0, \mathbf{x}_0) \in R}$

and $\mathbb{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ is continuous on R and

$\dfrac{\partial f_i}{\partial x_j}$ are continuous on R for $i, j = 1, \ldots, n$

Then there is some $r > 0$ so that the IVP $\begin{cases} \mathbf{x}' = \mathbb{F}(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$

has a unique solution for $t_0 - r < t < t_0 + r$

Example 3.3.1

$$\mathbb{X}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbb{X}$$

$$\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases} \qquad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad L\mathbb{X} = \mathbb{X}' - A\mathbb{X}$$

LINEAR CONSTANT COEFFICIENT SYSTEM

(HOMOGENEOUS)

$$L\mathbb{X} = 0$$

IMPORTANT FACTOID

If $(\lambda, \mathbb{V})$ is an eigenvalue/eigenvector pair, (for the constant matrix A)

then $\underline{\mathbb{X}(t) = c e^{\lambda t}\mathbb{V}}$ is a solution of $\mathbb{X}' = A\mathbb{X}$.

A straight line solution

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix},$$ ← Find eigenvalues/eigenvectors.

Wait! Why?

$$\mathbb{X}(t) = c e^{\lambda t}\mathbb{V}$$

constant

$$\mathbb{X}' = c\lambda e^{\lambda t}\mathbb{V}$$

matrix mult. is linear

$$= c e^{\lambda t} A\mathbb{V} = A(c e^{\lambda t}\mathbb{V})$$

$$= A\mathbb{X}.$$

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \lambda - 2 = 0$$

$$(\lambda+1)(\lambda-2) \qquad \lambda = \frac{1 \pm \sqrt{1+8}}{2}$$

$$\lambda = -1 \text{ or } 2 \quad \text{(eigenvalues)}$$

<u>eigenvectors</u>

<u>$\lambda = -1$</u> : $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \rightarrow$ $4v_1 - 2v_2 = 0$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

<u>one solution</u>: $\mathbf{x}(t) = c\, e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

<u>**Def'n**</u> If $\mathbb{X} = \mathbb{X}(t)$ is a (particular) solution, (of any autonomous system of ODE's $\mathbb{X}' = \mathbb{F}(\mathbb{X})$)

$$\{\mathbb{X}(t) : t \in \mathbb{R}\} \subseteq \text{phase space}$$

is called an <u>orbit</u> or <u>trajectory</u>.

<u>Generalized Factoid</u>: If A is a constant matrix and $(\lambda_1, \mathbb{V}_1), (\lambda_2, \mathbb{V}_2), \ldots, (\lambda_n, \mathbb{V}_n)$ are eigenvalue/eigenvector pairs <u>with</u>

$$\{\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_n\} \text{ a basis for } \mathbb{R}^n,$$

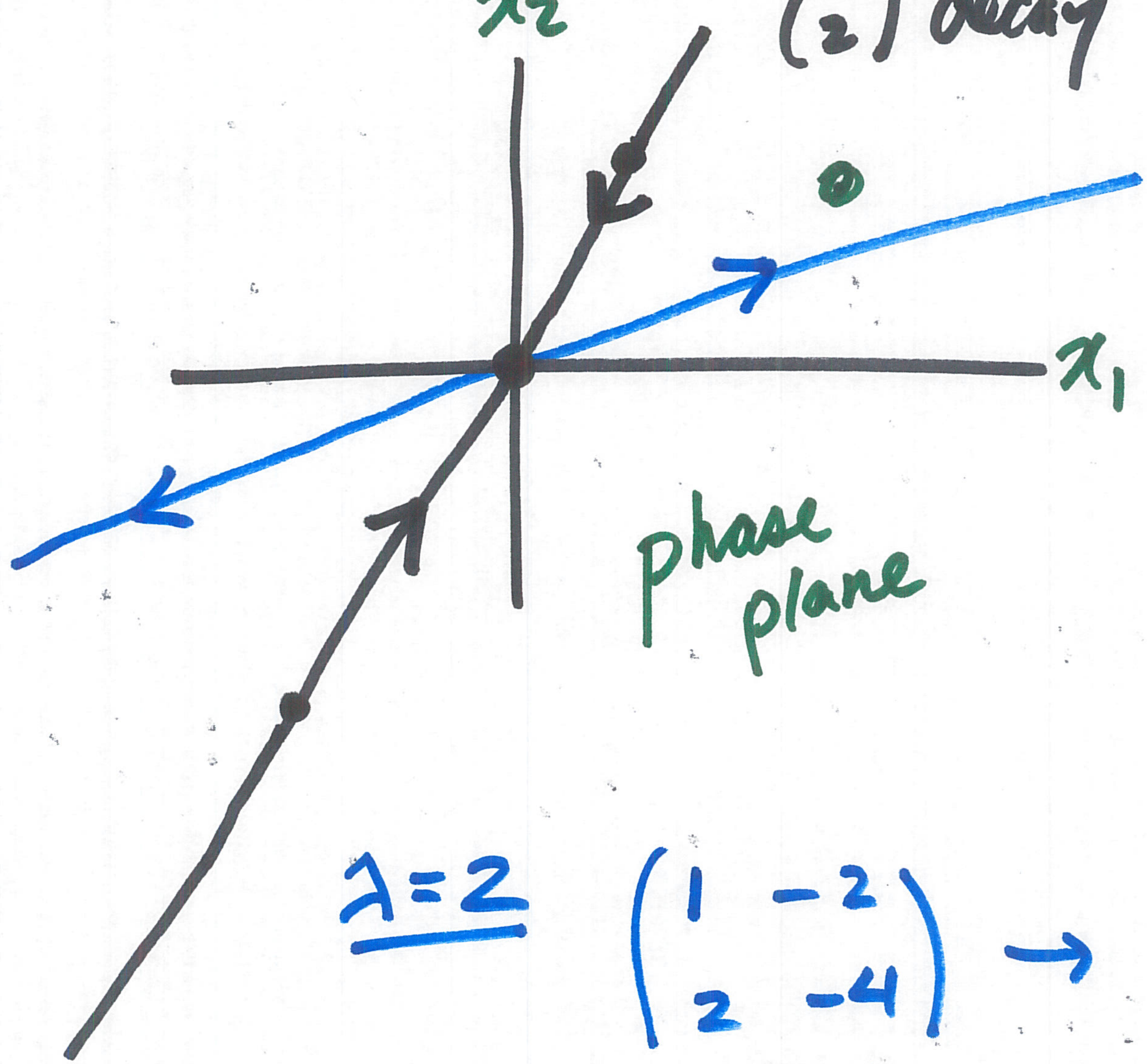
Then the general solution for $\mathbb{X}' = A\mathbb{X}$ is

$$\mathbb{X}(t) = c_1 e^{\lambda_1 t}\mathbb{V}_1 + c_2 e^{\lambda_2 t}\mathbb{V}_2 + \cdots + c_n e^{\lambda_n t}\mathbb{V}_n$$

$$= \sum_{j=1}^{n} c_j e^{\lambda_j t}\mathbb{V}_j$$

$$\begin{cases} x_1 = ce^{-t} \\ x_2 = 2ce^{-t} \end{cases} \qquad \mathbb{X}(t) = ce^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$\begin{pmatrix}1\\2\end{pmatrix}$ decay



$\mathbb{X}(0) = \begin{pmatrix}1\\2\end{pmatrix}$

$c = 1$

$$\begin{cases} \mathbb{X}(0) = \begin{pmatrix}0\\0\end{pmatrix} \\ c = 0 \end{cases}$$

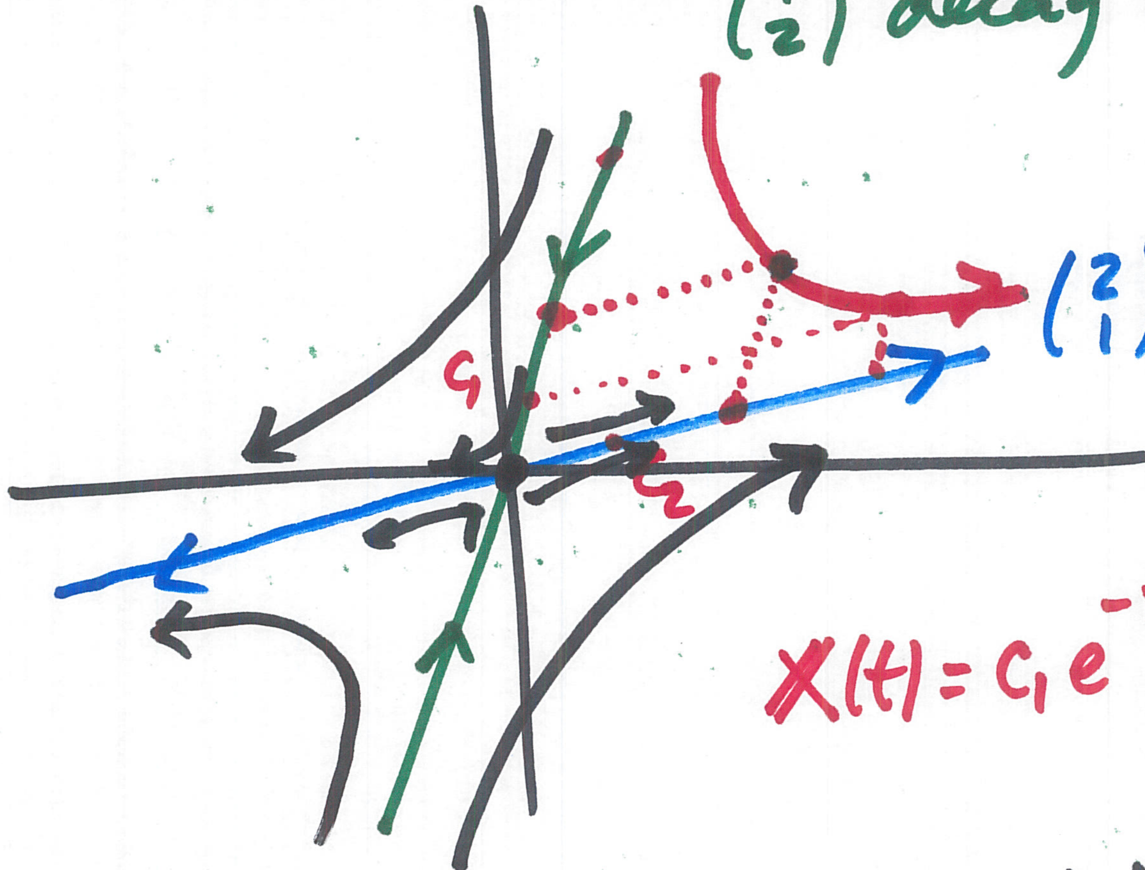
$\mathbb{X}(0) = \begin{pmatrix}-2\\-4\end{pmatrix}$

$c = -2$

$\underline{\lambda = 2}$ $\quad \begin{pmatrix}1 & -2\\2 & -4\end{pmatrix} \to \begin{pmatrix}2\\1\end{pmatrix}$

$\mathbb{X}(t) = ce^{2t}\begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}1\\2\end{pmatrix}$ decay (-1)



$\begin{pmatrix}2\\1\end{pmatrix}$ growth (2)

$$X(t) = c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$$

phase portrait (saddle).

(unstable)

3.3.25

$$y'' + 7y' + 10y = 0.$$

2nd order linear homogeneous

$$Ly = y'' + 7y' + 10y.$$

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = -10x_1 - 7x_2 \end{cases} \qquad A = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}$$