Math 2552, Final Exam (practice) Name (10 points) ____________________

1. (a) (10 points) Solve the IVP

$$\begin{cases} y' - y^2 - 1 = 0 \\ y(3) = 2. \end{cases}$$

(b) (10 points) Find the general solution of the ODE $y'' = 1 + (y')^2$.

Name and section: ____________________

Solution:

(a) Integrating $y'/(y^2+1) = 1$, we find $\tan^{-1} y - \tan^{-1} 2 = x - 3$ or $\tan^{-1} y = x + c$. In this case,

$$y = \tan\left(x - 3 + \tan^{-1} 2\right).$$

(b) Setting $v = y'$ we have from part (a) that $y' = v = \tan(x+c)$ where c is an arbitrary constant. Integrating again, we get

$$y(t) = -\ln[\cos(x+c)] + d \qquad \text{where } c \text{ and } d \text{ are arbitrary constants.}$$

Name and section: ____________________

2. (4.5.18,19)

   (a) (10 points) Find the general solution of the ODE $2y' + y = 4$.

   (b) (10 points) Solve the IVP $y'' + 4y = t^2 + 3e^t$, $y(0) = 0$, $y'(0) = 2$.

Name and section: ____________________

Solution:

(a) This is a linear first order equation $y' + y/2 = 2$:

$$\left(e^{t/2}y\right)' = 2e^{t/2}.$$

Integrating, we get $e^{t/2}y = 4e^{t/2} + c$, so the general solution is

$$y = 4 + ce^{-t/2} \qquad \text{where } c \text{ is an arbitrary constant.}$$

(b) $y_h = a\cos 2t + b\sin 2t$, and $y_p = At^2 + Bt + C + De^t$ gives

$$2A + De^t + 4At^2 + 4Bt + 4C + 4De^t = t^2 + 3e^t.$$

It follows that $2A + 4C = 0$, $5D = 3$, $A = 1/4$, and $B = 0$. Therefore,

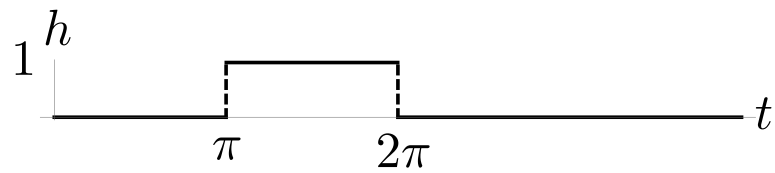
$$y = a\cos 2t + b\sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

From the initial conditions $0 = y(0) = a - 1/8 + 3/5$, so $a = -19/40$. Also, $2 = y'(0) = 2b + 3/5$, so $b = 7/10$. The solution is

$$y = -\frac{19}{40}\cos 2t + \frac{7}{10}\sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

Name and section: ____________________

3. (20 points) (5.6.2) Solve the IVP $y'' + 2y' + 2y = h(t)$, $y(0) = 5$, $y'(0) = 4$ where h is a solitary square wave as indicated below.



Name and section: ___________________

Solution: Notice that $h(t) = H(t-\pi) - H(t-2\pi) = u_\pi(t) - u_{2\pi}(t)$ where $H(t-c) = u_c(t)$ is the Heaviside function which turns on at time $t = c$. Thus, the Laplace transform of the IVP is

$$(s^2 + 2s + 2)Y - 5s - 4 - 10 = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

We have used rules number 36 and number 25 from the table. Solving for Y, we get

$$\begin{aligned} Y &= 5\frac{s}{(s+1)^2+1} + \frac{14}{(s+1)^2+1} + \frac{e^{-\pi s}}{s[(s+1)^2+1]} - \frac{e^{-\pi s}}{s[(s+1)^2+1]} \\ &= 5\frac{s+1}{(s+1)^2+1} + \frac{14-5}{(s+1)^2+1} + \frac{e^{-\pi s}}{s[(s+1)^2+1]} - \frac{e^{-\pi s}}{s[(s+1)^2+1]} \\ &= 5\mathcal{L}[e^{-t}\cos t] + 9\mathcal{L}[e^{-t}\sin t] + \frac{e^{-\pi s}}{s[(s+1)^2+1]} - \frac{e^{-\pi s}}{s[(s+1)^2+1]}. \end{aligned}$$

Using the method of partial fractions, we have

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{(s+1)^2+1)}$$

where $as^2 + 2as + 2a + bs^2 + cs = 1$ so that $a + b = 0$, $2a + c = 0$, and $2a = 1$ giving $a = 1/2$ and $bs + c = -(s+2)/2$.

Using rules number 7, number 8, and number 29 (or alternatively rules number 19 and number 20) on the table, we see

$$\frac{1}{s(s^2+2s+2)} = \mathcal{L}\left[\frac{1}{2}\left(1 - e^{-t}\cos t - e^{-t}\sin t\right)\right].$$

Finally, using rule number 27 to obtain the inverse, we have

$$\begin{aligned} y = 5e^{-t}\cos t + 9e^{-t}\sin t &+ \frac{H(t-\pi)}{2}\left(1 - e^{-(t-\pi)}\cos(t-\pi) - e^{-(t-\pi)}\sin(t-\pi)\right) \\ &- \frac{H(t-2\pi)}{2}\left(1 - e^{-(t-2\pi)}\cos(t-2\pi) - e^{-(t-2\pi)}\sin(t-2\pi)\right) \\ = 5e^{-t}\cos t + 9e^{-t}\sin t &+ \frac{H(t-\pi)}{2}\left(1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right) \\ &- \frac{H(t-2\pi)}{2}\left(1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right). \end{aligned}$$

Name and section: ____________________

4. (a) (15 points) Use Laplace transforms to model the motion of an undamped oscillator of unit mass and resonant frequency π in the following situation:

   - The oscillator is initially in motion with velocity 2 while passing through the equilibrium position (at time $t = 0$).
   - The mass experiences an impulse of magnitude 2 in the positive/upward direction at time $t = 3$.

   (b) (5 points) Find the limiting amplitude of the motion as $t \to \infty$.

Name and section: ______________________

Solution:

(a) Let us model this system using the operator $L[y] = y'' + \pi^2 y$. Given the initial values of position and velocity, we can translate the inital state of the motion into the Laplace transform space as

$$(s^2 + \pi^2)Y - 2.$$

The impulse on this motion is modeled in Laplace transform space by the algebraic problem

$$(s^2 + \pi^2)Y - 2 = 2e^{-3s}$$

which has solution

$$Y = \frac{2}{s^2 + \pi^2} + \frac{2e^{-3s}}{s^2 + \pi^2}.$$

Applying the inverse Laplace transform, we obtain a function y describing the motion:

$$y = \frac{2}{\pi}\left(\sin \pi t + H(t-3) \sin \pi(t-3)\right).$$

Since $\sin \pi(t-3) = -\sin \pi t$, this becomes

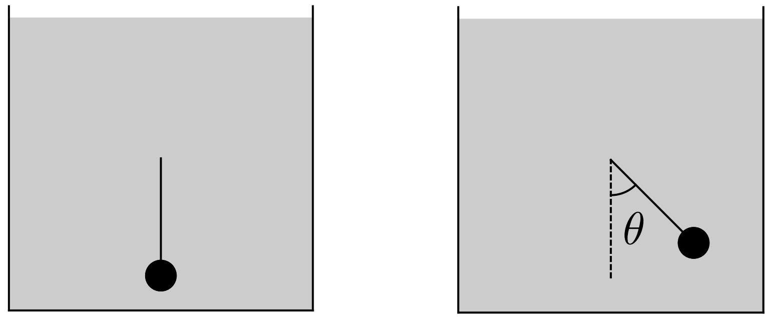
$$y = \begin{cases} 2 \sin \pi t/\pi, & 0 \le t \le 3 \\ 0, & t \ge 3. \end{cases}$$

(b) From the second formula above, it is clear that

$$\lim_{t \nearrow \infty} y(t) = y(3) = 0.$$

Name and section: ____________________

5. A pendulum is submerged in a damping fluid as indicated in the figure



and is modeled by the ODE

$$\theta'' + \alpha \tan^{-1} \theta' + k \sin \theta = 0 \tag{1}$$

where θ is the displacement angle and α and k are positive constants.

(a) (10 points) Setting $\omega = \theta'$, obtain a system of two first order ODEs that is equivalent to (1).

(b) (10 points) Find all equilibrium points for the system

$$\begin{pmatrix} \theta_* \\ \omega_* \end{pmatrix} =$$

Name and section: ____________________

Solution:

(a) The system is

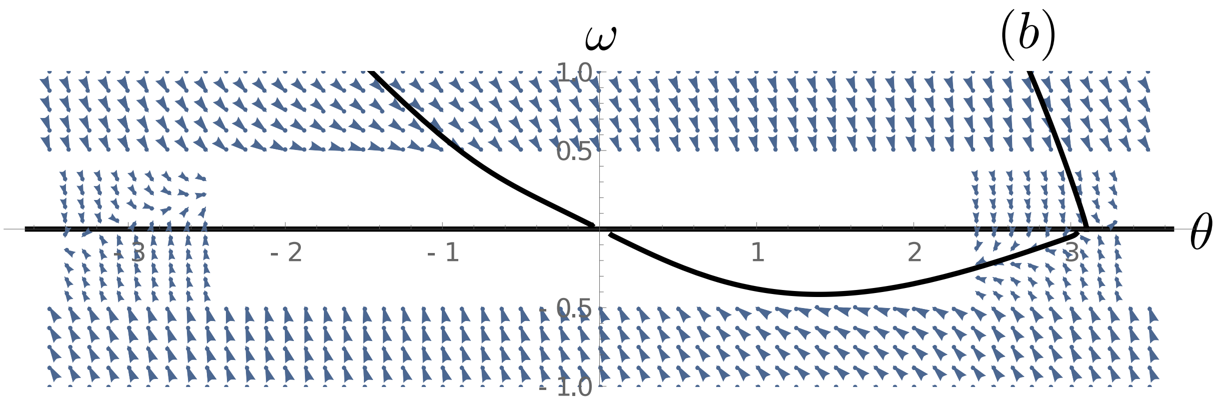
$$\begin{cases} \theta' = \omega \\ \omega' = -k \sin\theta - \alpha \tan^{-1}\omega. \end{cases}$$

(b)

$$\begin{pmatrix} \theta_* \\ \omega_* \end{pmatrix} = \begin{pmatrix} j\pi \\ 0 \end{pmatrix} \qquad \text{for } j = 0, \pm 1, \pm 2, \ldots.$$

Name and section: ________________

6. Consider the system you obtained in problem 5. A portion of the direction field for this system in a case where $\alpha > 2\sqrt{k}$ is shown. Also, three numerically calculated orbits are shown.



(a) (10 points) Linearize the system at the equilibrium point corresponding to the position with the pendulum hanging straight down (as shown on the left in the previous problem). Assuming $\alpha > 2\sqrt{k}$ draw the phase portrait of the linearized system.

(b) (10 points) Fill in more orbits on the direction field above to indicate clearly the bahavior of the system in this case. Explain the significance of the separatrix labeled (b).

Name and section: ____________________

Solution:

(a) The linearized system at the equilibrium (θ_*, ω_*) is

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k\cos\theta & -\alpha/(1+\omega^2) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

At $(\theta_*, \omega_*) = (0,0)$, this becomes

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k & -\alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

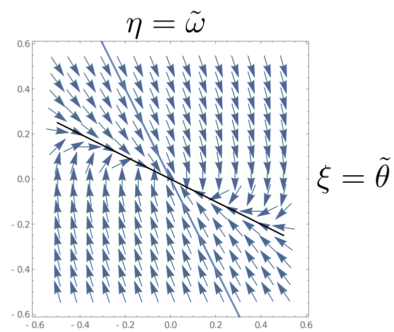
The matrix has characteristic equation $\lambda^2+\alpha\lambda+k=0$. The characteristic roots are

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2-4k}}{2}.$$

In the case under consideration, $\alpha > 2\sqrt{k}$, there are two distinct negative eigenvalues $\lambda_1 < \lambda_2 < 0$. For the more negative eigenvalue, the corresponding eigenvector $(v_1, v_2)^T$ satisfies

$$v_2 = \lambda_1 v_1.$$

Similarly, for the less negative eigenvalue the eigenvector lies along a direction with $v_2 = \lambda_2 v_2$. Since most orbits will be asymptotic to the weak decay direction, the phase portrait for the linear system looks like this:



(b) There are saddles at the odd multiples of π with the unstable separatrix limiting to the adjacent stable equilibrium. The crucial observation here is that the numerically calculated orbit in the second quadrant is clearly different from the numerically calculated orbit/separatrix connecting $(\pi,0)$ to $(0,0)$ (and not only different but also not symmetric to it with respect to the orgin, but they have the same limiting tangent line at $(0,0)$. This means they must both be limiting to the weak decay direction. In particular, the strong decay direction/separatrix at $(0,0)$ may be drawn in with a more negative slope. Once this is accomplished, the rest of the phase portrait is easy to fill in.

The stable separatrix (b) shown limiting to $(\pi,0)$ separates the orbits/initial values which tend to the equilibrium at $(\theta_*, \omega_*) = (0,0)$ from those which tend to an equilibrium having displacement a higher even multiple of π, for example those limiting to $(\theta_*, \omega_*) = (2\pi, 0)$.