

Homework 6, Analysis I

MATH 4317

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1. Rosenlicht page 90-95 Problems 17,18,23,27,29,32,33,39,40.
2. (length) A *simple curve* is the range of a function $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ which is continuous and satisfies $\gamma(s) = \gamma(t)$ only if $\{s, t\} \subset \{0, 1\}$.
 - (a) Give examples of five essentially different simple curves whose length *you know* how to compute. (Compute their lengths.)
 - (b) Show that

$$\text{length}(\gamma) = \sup \left\{ \sum_{j=1}^k |\gamma(a_j) - \gamma(a_{j-1})| : 0 = a_1 < a_2 < \cdots < a_k = 1 \right\}$$

is well defined, in the sense that it does not depend on the parameterization γ .

- (c) Show that the expression given in the previous part of this exercise gives correct lengths for the five curves you gave in the first part.
- (d) Is it true that for a given simple curve, any sequence of subdivisions/partitions $\mathcal{P}_\ell = \{0 = a_1^\ell < a_2^\ell < \cdots < a_k^\ell = 1\}$, whose *norms* $\max_j |a_j^\ell - a_{j-1}^\ell|$ tend to zero as ℓ tends to ∞ satisfies

$$\lim_{\ell \rightarrow \infty} \sum_{j=1}^k |\gamma(a_j^\ell) - \gamma(a_{j-1}^\ell)| = \text{length}(\gamma)?$$

- (e) What if we add the condition that $\mathcal{P}_{\ell+1} \subset \mathcal{P}_\ell$, i.e., that $\mathcal{P}_{\ell+1}$ is a *refinement* of \mathcal{P}_ℓ ?

3. (area) A *simple surface* is the range of a function $X : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^n$ which is continuous and whose restriction to $(0, 1) \times (0, 1)$ is injective.

Show that the analogue of the construction given in the previous problem for length does not work for area as follows:

- (a) Let $X(s, t) = (\cos 2\pi s, \sin 2\pi s, t) \in \mathbb{R}^3$ for $(s, t) \in [0, 1] \times [0, 1]$. Show that this defines a simple surface.
- (b) Draw a picture of this surface and compute its area.

- (c) Show that for any number $M \geq 2\pi$ and any $\epsilon > 0$, one may subdivide $[0, 1] \times [0, 1]$ into triangles of side length less than ϵ so that the sum of the areas of the corresponding triangles determined by the images of the vertices in \mathbb{R}^3 is within ϵ of M . (Three points in \mathbb{R}^3 determine a unique triangle.)
- (d) Draw a picture of your construction.