1. (25 points) (6C) Define *supremum*.

Give an example of a set of rational numbers which is bounded above but does not have a rational supremum.

**Solution:** The *supremum* of a nonempty set of real numbers $A$ is an upper bound for $A$, that is, a number $M$ such that $a \leq M$ for every $a \in A$, with the following property:

If $B$ is any upper bound for $A$, then $B \geq M$.

Example:

$$A = \{q \in \mathbb{Q} : q^2 < 2\}$$

This set contains $1/2$ and is clearly bounded above by, for example, 2. But $\text{sup } A = \sqrt{2} \notin \mathbb{Q}$.

2. (25 points) (8G) Define the term *norm*.

Prove that $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ defines a norm on $\mathbb{R}^n$.

**Solution:** Given a vector space $V$, a *norm* is a real valued function on $V$ with the following properties:

1. $\|v\| \geq 0$ with equality only if $v = 0$.
2. $\|v + w\| \leq \|v\| + \|w\|$.
3. $\|cv\| = |c|\|v\|$.

Clearly, the function on $x$ defined above has domain the vector space $\mathbb{R}^n$ and is real valued. The function is clearly nonnegative and positive definite since $\|x\| = 0$ implies each component is zero. In fact, $|x_j| = 0$ for each $j$ and we know that absolute value is a norm on $\mathbb{R}$. Therefore $x_j = 0$ for each $j$ as claimed. Thus, $x = 0$, and $\|\cdot\|$ is positive definite.

Next, we see that $\|x + y\| = \sum |x_j + y_j| \leq \sum |x_j| + |y_j| = \|x\| + \|y\|$ since absolute value satisfies the triangle inequality on $\mathbb{R}$.

Finally, $\|cx\| = \sum |cx_j| = |c| \sum |x_j|$. Again, this is because absolute value is a norm on $\mathbb{R}$. 
3. (25 points) (9L) Define the terms *closed set* and *closure*.

Prove that the closure of a set is a closed set.

**Solution:** A set is *closed* if its complement is open.

The *closure* of a set $A$, denoted by $\bar{A}$, is the intersection of all closed sets containing $A$. In symbols:

$$\bar{A} = \bigcap_{B \supseteq A} B.$$ 

To see that the closure is closed, note that

$$\bar{A}^c = \bigcap_{B \supseteq A} B^c.$$ 

Since the sets $B$ above are closed, the sets $B^c$ are open. And since the union of open sets is open, we see that the complement of $\bar{A}$ is open. This means $\bar{A}$ is closed.

4. (25 points) (11A) Define the term *compact*.

Prove directly from the definitions (without using the Heine-Borel Theorem) that a closed subset of a compact set is compact.

**Solution:** A set $K$ is compact if any open cover of $K$ has a finite subcover.

Let $K$ be a compact set and let $C$ be a closed subset of $K$. Then notice that $U_0 = C^c$ is an open set. Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be any open cover of $C$. Then $\{U_\alpha\} \cup \{U_0\}$ is an open cover of $K$. Since $K$ is compact, this cover has a finite subcover:

$$\{U_0, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}.$$ 

We claim that $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}$ is a finite cover of $C$. In fact, if $x \in C$, then $x \in K \setminus U_0$. Therefore $x$ must be in $\cup U_\alpha$. Therefore, $C \subseteq \cup U_\alpha$, and $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}$ is a cover. It is therefore a finite open subcover of $C$, and $C$ is compact.