Math 4441, Extra Problems: Differential Geometry (practice)

These questions were drafted for the final exam in 2015 but were omitted.

1. (regular surfaces/first fundamental form) Let $\mathcal{S}$ be a regular surface parameterized locally by X with $X(u_0, v_0) = p \in \mathcal{S}$. Show there exists a local coordinate $\tilde{X}$ which parameterized $\mathcal{S}$ near p with $\tilde{X}(0,0) = p$ and such that the coordinate tangent vectors $\tilde{X}_u(0,0)$ and $\tilde{X}_v(0,0)$ are orthogonal at p.

Solution: An affine change of coordinates of the form

$$\tilde{X}(\xi, \eta) = X(u_0 + \xi + c\eta, v_0 + \eta)$$

satisfies

$$\tilde{X}_\xi(0,0) = X_u(u_0, v_0) \qquad \text{and} \qquad \tilde{X}_\eta(0,0) = cX_u(u_0, v_0) + X_v(u_0, v_0).$$

Thus,

$$\tilde{E}(0,0) = E(u_0, v_0), \qquad \tilde{F}(0,0) = cE(u_0, v_0) + F(u_0, v_0), \quad \text{and} \quad \tilde{G}(0,0) = c^2 E(u_0, v_0) + 2cF(u_0, v_0) +$$

Therefore, by taking $c = -F(u_0, v_0)/E(u_0, v_0)$, we get $\tilde{F}(0,0) = 0$, and

$$\tilde{G}(0,0) = -F(u_0, v_0)^2/E(u_0, v_0) + G(u_0, v_0) = \frac{E(u_0, v_0)G(u_0, v_0) - F(u_0, v_0)^2}{E(u_0, v_0)}.$$

The condition on F is what we want, and the only thing we need to verify in order to know we have a valid reparameterization is that $\tilde{G}$ does not vanish. Since the value of $\tilde{G}(u_0, v_0)$ is positive, we can restrict to a smaller neighborhood if necessary to get an open set around $(0,0)$ where $\tilde{X}$ is a valid coordinate.

2. (curvature calculation) The catenoid is parameterized by $X(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$. The helicoid is parameterized on the same domain by $\tilde{X}(u, v) = (\sinh v \cos u, \sinh v \sin u, u)$. This problem concerns the deformation

$$Y(u, v; t) = \cos t\, X(u, v) + \sin t\, \tilde{X}(u, v) \quad \text{for} \quad 0 \le t \le \pi/2.$$

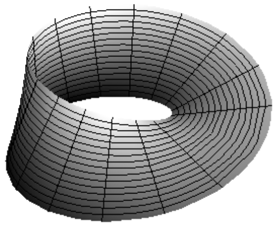
Calculate the mean curvature of the regular surface parameterized by $Y(u, v) = Y(u, v; t)$.

Solution:

3. (25 points) (surfaces; area; 2-6.3) A sculpture in the shape of a Möbius strip has parameterization

$$X(u, v) = (2\cos(u) + v\cos(u/2)\cos(u), 2\sin(u) + v\cos(u/2)\sin(u), v\sin(u/2))$$

where $u \in \mathbb{R}$, $-1 \leq v \leq 1$, and the physical dimensions are given in yards. If paint covers 250 ft^2/gal, how many quarts should be purchased to paint this sculpture?



(Hints: There are three feet in a yard and four quarts in a gallon.)