Math 4441, Exam 1: Curves (practice) Name and section: ____________________

1. (25 points) (1-2.1) Find a clockwise parameterization of the circle $x^2 + y^2 = 4$.

Solution: The standard (ccw) parameterization of a circle is

$$\gamma_{cc}(\theta) = (r\cos\theta, r\sin\theta).$$

Replace θ with $-\theta$ to get a clockwise parameterization:

$$\gamma(\theta) = (r\cos\theta, -r\sin\theta).$$

2. (25 points) (1-4.3) Determine the angle of intersection of the planes determined by $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.

Solution: The angle between the planes is the angle between their normals. The normals are $(5, 3, 2)$ and $(3, 4, -7)$. The cosine of the angle between these vectors is their dot product divided by the product of their norms:

$$\cos\theta = \frac{15 + 12 - 14}{\sqrt{25 + 9 + 4}\sqrt{9 + 16 + 49}} = \frac{13}{\sqrt{(38)(74)}} = \frac{13}{2\sqrt{703}}.$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{703}}\right).$$

3. (25 points) (1-5.1) Find the osculating plane of the helix parameterized by

$$\alpha(s) = (3\cos(s/5), 3\sin(s/5), 4s/5).$$

Solution: Note that

$$\alpha'(s) = (-3\sin(s/5)/5, 3\cos(s/5)/5, 4/5).$$

Thus,

$$|\alpha'(s)| = 1,$$

and s is an arclength parameter. Therefore, the principal normal is given by

$$\alpha''(s) = (-3\cos(s/5)/25, -3\sin(s/5)/25, 0).$$

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The osculating plane is spanned by the tangent vector and the principle normal to the curve, so a normal to the plane is given by

$$N = \alpha' \times (\cos(s/5), \sin(s/5), 0) = (4\sin(s/5)/5, 4\cos(s/5)/5, -3/5).$$

Thus, the osculating plane is $\{p \in \mathbb{R}^3 : (p - \alpha(s)) \cdot N = 0\} =$

$\{(x, y, z) \in \mathbb{R}^3 : 4(x - 3\cos(s/5))\sin(s/5) + 4(y - \sin(s/5))\cos(s/5) - 3(z - 4s/5) = 0\}$.

4. (25 points) (1-7.3) Compute the curvature of the ellipse with parameterization

$$r(t) = (3\cos t, 2\sin t).$$

Solution:

$$r'(t) = (-3\sin t, 2\cos t) \quad \text{and} \quad |r'(t)| = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{5\sin^2 t + 4}.$$

Therefore,

$$\gamma(s) = r(\tau(s))$$

is a parameterization by arclength where $\tau = \tau(s)$ is determined by

$$s = \int_0^\tau \sqrt{5\sin^2 t + 4}\, dt.$$

This means that

$$\dot{\tau} = \frac{1}{\sqrt{5\sin^2 \tau + 4}},$$

and

$$k(\tau) = |\ddot{\gamma}(s)|.$$

Therefore,

$$\dot{\gamma}(s) = r'(\tau)\dot{\tau} = \frac{(-3\sin\tau, 2\cos\tau)}{\sqrt{5\sin^2\tau + 4}},$$

Thus,

$$\ddot{\gamma}(s) = \frac{(-3\cos\tau, -2\sin\tau)}{5\sin^2\tau + 4} - \frac{5\sin\tau\cos\tau(-3\sin\tau, 2\cos\tau)}{(5\sin^2\tau + 4)^2}.$$

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Finally,

$$\begin{aligned}
k(\tau) &= |\ddot{\gamma}(s)| \\
&= \frac{\sqrt{5\cos^2\tau + 4 - 2\frac{25\sin^2\tau\cos^2\tau}{5\sin^2\tau+4} + \frac{25\sin^2\tau\cos^2\tau}{5\sin^2\tau+4}}}{5\sin^2\tau + 4} \\
&= \frac{\sqrt{5\cos^2\tau + 4 - \frac{25\sin^2\tau\cos^2\tau}{5\sin^2\tau+4}}}{5\sin^2\tau + 4} \\
&= \frac{\sqrt{(5\cos^2\tau + 4)(5\sin^2\tau + 4) - 25\sin^2\tau\cos^2\tau}}{(5\sin^2\tau + 4)^{3/2}} \\
&= \frac{6}{(5\sin^2\tau + 4)^{3/2}}.
\end{aligned}$$