**Math 4441, Exam 2: Surfaces (practice)** Name and section: ________________

1. (25 points) (2-2.2) Is the set $\{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } x^2 + y^2 \leq 1\}$ a regular surface?

**Solution:** This is not a regular surface because a neighborhood of a boundary point like $(1, 0, 0)$ is not homeomorphic to an open set of $\mathbb{R}^2$ with the point corresponding to $(1, 0, 0)$ in the interior.

A justification of the assertion that the two sets in question above are not homemorphic goes roughly like this: Assume there is a homemorphism $h$ with domain a neighborhood $U$ of $(1, 0, 0)$ in

$$M = \{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } x^2 + y^2 \leq 1\}$$

and image a neighborhood of a point, say $(0, 0) \in \mathbb{R}^2$ with $h(1, 0, 0) = (0, 0)$. The image neighborhood $h(U)$ contains a circle $\Gamma = \partial B_r(0, 0)$. The inverse image $C = h^{-1}(\Gamma)$ of this circle is a simple closed curve in $N$. Also, by taking $\Gamma$ small enough, which we can do, we may assume $C \subset B_\rho(1, 0, 0) \cap M$ for some $\rho > 0$. Two things can then be seen pretty easily. First, there is a homotopy which shrinks $C$ to a point within $N \backslash \{(1, 0, 0)\}$. That is, there is a continuous map $\phi : [0, 1] \times C \to N \backslash \{(1, 0, 0)\}$ with $\phi(0, t)$ a regular parameterization of $C$ and $\phi(1, t)$ constant. Using the homeomorphism, we get a homotopy $\psi = h \circ \phi$ with image in $B_r(0, 0) \backslash \{(0, 0)\}$ which shrinks $\Gamma$ to a point. Second, each image $\psi(s, C)$ has at least one point with a positive $x$ coordinate and at least one point with a negative $x$ coordinate. Since these points must be different, we get a contradiction for $s = 1$.

2. (25 points) (2-3.3) Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

**Solution:** A global coordinate on the paraboloid $P$ is given by $X(u, v) = (u, v, u^2 + v^2)$, and the plane $\mathbb{R}^2$ has a trivial global coordinate $Y(x, y) = (x, y)$.

Consider the function $\phi : P \to \mathbb{R}^2$ by $\phi(p_1, p_2, p_3) = (p_1, p_2)$. First note that the composition $\phi \circ X(u, v) = (u, v)$ is the identity map. So it's clear that this function is smooth. On the other hand, given a point $(x, y)$ in the plane, there is exactly one point $p = (p_1, p_2, p_3) \in P$ with $\phi(p) = (x, y)$, namely, $(x, y, x^2 + y^2)$. Thus, $\phi$ has in inverse. Also, $X^{-1} \circ \phi^{-1}(x, y) = (x, y)$. This is also the identity map and is smooth. Thus, $\phi$ is a diffeomorphism.

3. (25 points) (2-4.2) Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$.

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**Solution:** Since $z^2 = x^2 + y^2 - 1$, we see that this surface is parameterized (in polar/cylindrical) coordinates by

$$X(\theta, z) = (\sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z)$$

on $[0, 2\pi] \times \mathbb{R}$. Note that the points in question corresponding to $z = 0$ lie on the unit circle in the $x, y$-plane. The tangent vectors here are

$$X_\theta = (-\sin\theta, \cos\theta, 0) \quad \text{and} \quad X_z = (0, 0, 1).$$

Thus, the plane at $(x, y, 0) = (\cos\theta, \sin\theta, 0)$ is

$$\{(\xi, \eta, \zeta) : (\xi - x, \eta - y, \zeta) \cdot (\cos\theta, \sin\theta, 0) = 0\} = \{(\xi, \eta, \zeta) : \xi\cos\theta + \eta\sin\theta = 1\}.$$

4. (25 points) (2-5.1) Compute the first fundamental form of the ellipsoid parameterized by

$$X(u, v) = (3\sin u\cos v, 4\sin u\sin v, 5\cos u).$$

**Solution:**

$$X_u = (3\cos u\cos v, 4\cos u\sin v, -5\sin u),$$

and

$$X_v = (-3\sin u\sin v, 4\sin u\cos v, 0).$$

Thus, the coefficients of the first fundamental form are given by

$$E = |X_u|^2 = \cos^2 u(9\cos^2 v + 16\sin^2 v) + 25\sin^2 u,$$

$$F = X_u \cdot X_v = 7\cos u\cos v\sin u\sin v,$$

and

$$G = |X_v|^2 = \sin^2 u(9\sin^2 v + 16\cos^2 v).$$

The first fundmental form of a tangent vector $aX_u + bX_v$ at $X(u, v)$ is therefore

$$\begin{aligned}\mathrm{I}\,(aX_u + bX_v) &= Ea^2 + 2Fab + Gb^2 \\ &= [\cos^2 u(9\cos^2 v + 16\sin^2 v) + 25\sin^2 u]a^2 \\ &\quad + 2[7\cos u\cos v\sin u\sin v]ab \\ &\quad + \sin^2 u(9\sin^2 v + 16\cos^2 v)b^2.\end{aligned}$$