Math 4441, Exam 3: Gauss Map (practice) Name/Section: ________________

1. (25 points) (3-2.8) Determine the region of the unit sphere covered by the Gauss map of the paraboloid $z = x^2 + y^2$.

Solution: A parameterization is given by $X(u, v) = (u, v, u^2 + v^2)$.

$$X_u = (1, 0, 2u) \quad \text{and} \quad X_v = (0, 1, 2v).$$

$$\sqrt{EG - F^2}N = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix} = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}.$$

This is the upward normal (Gauss map). It covers the open upper half sphere. To see this, first note that the third component of

$$N = (-2u, -2v, 1)/\sqrt{1 + 4u^2 + 4v^2} \tag{1}$$

is positive. This means all images are in $\mathbb{S}^2_+ = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z > 0\}$. On the other hand, given any $(x, y, z) \in \mathbb{S}^2_+$, we can take $u = -x/(2\sqrt{1 - x^2 - y^2})$ and $v = -y/(2\sqrt{1 - x^2 - y^2})$, we will get $N(u, v) = (x, y, z)$. (Just plug these values into (1) and use the fact that $z = \sqrt{1 - x^2 - y^2}$.)

2. (25 points) (3-2.15) Given that two surfaces intersect along a regular curve at a constant angle and that the curve is a curvature line on one of the surfaces, prove that the intersection curve is a curvature line on the other surface.

Solution: We can parameterize the preimage of the curve in the coordinates (at least locally), so that $u = u_j(t)$, $v = v_j(t)$ and the curve is given by $\gamma(t) = X_j(u(t), v(t))$ for $j = 1, 2$. The condition that the curves meet at a constant angle is

$$N_1 \cdot N_2 = \text{ constant}$$

along the curve (at corresponding points). The condition that curve be a curvature line along one of the surfaces is

$$dN(\gamma') = \frac{d}{dt}N_j = \lambda\gamma'$$

for some (function) λ.

Now, say you have a curvature line on one surface, and the surfaces meet at a constant angle. If that angle is 0 or π so that $N_1 \cdot N_2 = \pm 1$, then $N_2 = \pm N_1$, and

$$N_2' = \pm N_1' = \pm\lambda\gamma',$$

and we are done. This means we can assume $N_1 \cdot N_2 \neq \pm 1$, and $\{\gamma', N_1, N_2\}$ forms a basis for $\mathbb{R}^3$.

Differentiating the angle condition we get

$$N_1' \cdot N_2 + N_1 \cdot N_2' = 0. \tag{2}$$

If the curvature line is on the first surface, then N_1' is tangent to the intersection curve, and lies in the tangent plane of the second surface in particular. Thus, the first term in (2) vanishes and we get

$$N_1 \cdot N_2' = 0.$$

This means N_2' is orthogonal to N_1. Differentiating the norm condition $N_2 \cdot N_2 = 1$, we find that N_2' is also orthogonal to N_2. This means N_2' is orthogonal to the two-dimensional subspace spanned by N_1 and N_2, and has no component in that subspace. In other words, N_2' is a multiple of γ'. Thus, $N_2' = \mu\gamma'$ for some function μ, and γ is a curvature line on the second surface.

For the reverse direction, we assume that γ is a curvature line on both surfaces. Thus, $N_1' = \lambda\gamma'$ and $N_2' = \mu\gamma'$. Differentiating $N_1 \cdot N_2$, we find

$$N_1' \cdot N_2 + N_1 \cdot N_2' = \lambda\gamma' \cdot N_2 + N_1 \cdot (\mu\gamma') = 0$$

since γ' is tangent to the surfaces and N_j is normal for $j = 1, 2$.

3. (25 points) (3-3.1) Compute the Gauss curvature and mean curvature of the hyperboloid $z = xy$.

Solution: As usual we use the graph parameterization $X(u, v) = (u, v, uv)$.

$$X_u = (1, 0, v) \quad \text{and} \quad X_v = (0, 1, u).$$

$$E = 1 + v^2,\ F = uv,\ G = 1 + u^2.$$

$$EG - F^2 = 1 + u^2 + v^2,$$

and

$$N = (-v, -u, 1)/\sqrt{1 + u^2 + v^2}.$$

$$X_{uu} = 0,\ X_{uv} = \mathbf{e}_3,\ X_{vv} = 0.$$

Thus, the coefficients of the second fundamental form are

$$e = X_{uu} \cdot N = 0,\ f = X_{uv} \cdot N = 1/\sqrt{1 + u^2 + v^2},\ \text{and } g = X_{vv} \cdot N = 0.$$

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Thus, the Gauss curvature is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1/(1 + u^2 + v^2)}{1 + u^2 + v^2} = \frac{-1}{(1 + u^2 + v^2)^2},$$

and the mean curvature (with respect to the upward normal) is

$$H = \frac{1}{2}\,\frac{eG - 2fF + Eg}{EG - F^2} = \frac{-2uv/\sqrt{1 + u^2 + v^2}}{2(1 + u^2 + v^2)} = \frac{-uv}{(1 + u^2 + v^2)^{3/2}}.$$

4. (25 points) (3-3.15) Give an example of a surface which has an isolated parabolic point.

Solution: The basic idea is that one point should look like a cylinder (to second order), that is it should have one zero curvature direction and one nonzero direction orthogonal to it. We know the graph of x^4 has an isolated point of zero curvature, so we can use this in one direction. Then we need something to "bend it up" in the other direction away from the origin. We can do this with a graph which looks like x^4 in one direction (only at the origin) and y^2 in the other:

$$X(u, v) = (u, v, u^4 + u^2v^2 + v^2).$$

This has a parabolic point at the origin, and the other points are all elliptic. To see this, compute the second fundamental form:

$$X_u = (1, 0, 4u^3 + 2uv^2), \quad X_v = (0, 1, 2u^2v + 2v),$$

and

$$N = (-4u^3 - 2uv^2, -2u^2v - 2v, 1)/\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}.$$

$$X_{uu} = (0, 0, 12u^2 + 2v^2), \quad X_{uv} = (0, 0, 4uv), \quad X_{vv} = (0, 0, 2u^2 + 2).$$

$$e = (12u^2 + 2v^2)/\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2},$$

$$f = 4uv/\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2},$$

$$g = (2u^2 + 2)/\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}.$$

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Given a tangent direction $aX_u + bX_v$, we know that

$$\begin{aligned} II(aX_u + bX_v) &= ea^2 + 2fab + gb^2 = ea^2 + gb^2 \\ &= \frac{(12u^2 + 2v^2)a^2 - 8uvab + (2u^2 + 2)b^2}{\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}} \\ &= \frac{(6u^2 + v^2)a^2 - 4uvab + (u^2 + 1)b^2}{2\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}} \\ &= \frac{6u^2a^2 + v^2a^2 - 4uvab + (u^2 + 1)b^2}{2\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}} \\ &= \frac{6u^2a^2 + (va - 2ub)^2 + (-3u^2 + 1)b^2}{2\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}}. \end{aligned}$$

Notice that the numerator in the expression above is strictly positive whenever $|u| < 1/3$ unless $b = 0$. This means there is at most one possible direction at each point near the origin with zero sectional curvature. Taking this direction and assuming it is nondegenerate, we have $a \neq 0$. Finally, for a zero curvature in this direction, we must have

$$\frac{6u^2a^2 + v^2a^2}{2\sqrt{1 + 16u^6 + 24u^4v^2 + 12u^2v^2 + 4v^2}} = 0.$$

This can only happen when $u = v = 0$, i.e., at the origin.