Math 4441, Exam 4: Intrinsic Geometry (practice)Name/Section: ________________

1. (25 points) (4-2.4) Define stereographic projection and show that stereographic projection is a conformal map.

Solution: Stereographic projection $\pi : \mathbb{S}^2\backslash\{(0,0,1)\} \to \mathbb{R}^2$ is given by

$$\pi(x, y, z) = (x, y)/(1 - z).$$

To see that this map is conformal, we recall that inverse stereographic projection provides coordinates on $\mathbb{S}^2_* = \mathbb{S}^2\backslash\{(0,0,1)\}$ given by

$$X(u, v) = (2u, 2v, u^2 + v^2 - 1)/(u^2 + v^2 + 1)$$

where $X : \mathbb{R}^2 \to \mathbb{S}^2_*$. Calculating the first fundamental form in these coordinates, we find

$$X_u = 2(-u^2 + v^2 + 1, -2uv, 2u)/(u^2 + v^2 + 1)^2$$

and

$$X_v = 2(-2uv, u^2 - v^2 + 1, 2v)/(u^2 + v^2 + 1)^2.$$

$$E = \frac{4}{(u^2 + v^2 + 1)^2}, \quad F = 0, \quad G = \frac{4}{(u^2 + v^2 + 1)^2}.$$

Thus,

$$I(aX_u + bX_v) = E(a^2 + b^2).$$

Notice that this says

Given a vector $(a, b) \in T_{(u,v)}\mathbb{R}^2$,

$$|(a, b)|^2 = |dX((a, b))|^2/E.$$

(The norm on the left is in $\mathbb{R}^2$*; the norm on the right is in* $\mathbb{R}^3$*.)*

Using the polarization identities

$$\langle \mathbf{v}, \mathbf{w}\rangle_p = (1/2)[\langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{w}, \mathbf{w}\rangle_p]$$

where $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^2_*$ and

$$\langle \mathbf{v}, \mathbf{w}\rangle = (1/2)[\langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{w}, \mathbf{w}\rangle]$$

for $\mathbf{v}, \mathbf{w} \in T_{(u,v)}\mathbb{R}^2$, we see that for $p = X(u, v)$

$$\begin{aligned}\langle \mathbf{v}, \mathbf{w}\rangle &= (1/(2E))[|dX(\mathbf{v} + \mathbf{w})|^2 - |dX(\mathbf{v})|^2 - |dX(\mathbf{w})|^2] \\ &= \langle dX(\mathbf{v}), dX(\mathbf{w})\rangle_p/E.\end{aligned}$$

Since $dX^{-1} = d\pi : T_p\mathbb{S}^2_* \to T_{\pi(p)}\mathbb{R}^2$, this means

$$\langle \mathbf{v}, \mathbf{w}\rangle_p = E\langle d\pi_p(\mathbf{v}), d\pi_p(\mathbf{w})\rangle$$

for $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^2_*$. This is the definition of conformality.

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2. (25 points) (4-2.12) Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be a cylinder. Find an isometry $\phi : C \to C$ such that the set of *fixed points*, i.e., $\{p \in C : \phi(p) = p\}$, contains exactly two points.

Solution: Rotation of the cylinder about the x-axis by an angle of $180°$ should accomplish what we need:

$$\phi(x, y, z) = (x, -y, -z).$$

Clearly, this map is one-to-one, onto, continuous, and has a continuous inverse. In fact, it is clear that the map is smooth as well as its inverse. Thus, ϕ is a diffeomorphism. Moreover, if $X = (X_1, X_2, X_3)$ is any parameterization, then comparing the parameterizations X and $\tilde{X} = \phi \circ X = (X_1, -X_2, -X_3)$, we see that

$$\tilde{E} = E, \quad \tilde{F} = F, \quad \text{and} \quad \tilde{G} = G.$$

Thus, given $\mathbf{v}, \mathbf{w} \in T_pC$, we can take $\mathbf{v}_0 = dX^{-1}(\mathbf{v})$ and $\mathbf{w}_0 = dX^{-1}(\mathbf{w})$. Then,

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle_p &= \langle dX(\mathbf{v}_0), dX(\mathbf{w}_0) \rangle_p \\ &= \langle d\tilde{X}(\mathbf{v}_0), d\tilde{X}(\mathbf{w}_0) \rangle_{\phi(p)}. \end{aligned}$$

This is true because the first fundamental forms of X and $\tilde{X}$ are the same. Finally, since $d\tilde{X} = d\phi \circ dX$, we see that

$$\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle d\phi(\mathbf{v}), d\phi(\mathbf{w}) \rangle_{\phi(p)}.$$

This means that ϕ is an isometry.

We need to check the fixed points: If $(x, -y, -z) = (x, y, z) \in C$, then clearly $y = z = 0$. Since also $x^2 + y^2 = 1$, we conclude that $x = \pm 1$, and there are exactly two such points $(\pm 1, 0, 0)$.

3. (25 points) (4-3.3) Let

$$X(u, v) = (u \cos v, u \sin v, \ln u)$$

and

$$\tilde{X} = (u \cos v, u \sin v, v).$$

Show that $\tilde{X} \circ X^{-1}$ is not an isometry.

Solution: If these parameterized surfaces were isometric, then they would have the same first fundamental form at corresponding points. In fact,

$$X_u = (\cos v, \sin v, 1/u), \quad X_v = (-u \sin v, u \cos v, 0),$$

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so that

$$E = 1 + 1/u^2, \quad F = 0, \quad G = u^2,$$

while

$$\tilde{X}_u = (\cos v, \sin v, 0), \quad \tilde{X}_v = (-u \sin v, u \cos v, 1),$$

and

$$\tilde{E} = 1, \quad \tilde{F} = 0, \quad \tilde{G} = u^2 + 1.$$

Thus, the first fundamental forms are not the same, and the surfaces can not be isometric.

4. (25 points) (4-3.4) Show that no neighborhood of a sphere may be isometrically mapped into a portion of the plane.

Solution: Were such an isometry to exist, the sphere and the plane would have the same Gauss curvature at corresponding points. However, the Gauss curvature of a sphere is a nonzero constant, and the Gauss curvature of a plane is constant zero. Consequently, there can be no such isometry.