Math 4441, Final Exam: Diff. Geom. (practice) Name/Section: ________________

1. (25 points) (1-5.14) Let $\gamma : (0,1) \to \mathbb{R}^3$ be a regular parameterized curve. If $|\gamma(t)| \leq 3$ for $0 < t < 1$, and $|\gamma(1/2)| = 3$, show that the curvature $k = k(1/2)$ at $\gamma(1/2)$ satisfies

$$k \geq \frac{1}{3}.$$

> **Solution:** Reparemeterize the curve by arclength, let the reparameterization be denoted by α, and assume that $\alpha(0) = \gamma(1/2)$. Since $|\alpha(s)|^2$ has a local max at $s = 0$, we have
>
> $$\frac{d}{ds}|\alpha(s)|^2\Big|_{s=0} = 0 \quad \text{and} \quad \frac{d^2}{ds^2}|\alpha(s)|^2\Big|_{s=0} \leq 0.$$
>
> These conditions imply
>
> $$\dot{\alpha}(0) \cdot \alpha(0) = 0 \quad \text{and} \quad |\dot{\alpha}(0)|^2 + \ddot{\alpha}(0) \cdot \alpha(0) \leq 0.$$
>
> Since $\dot{\alpha}$ is a unit vector, the second inequality becomes
>
> $$\ddot{\alpha}(0) \cdot \alpha(0) \leq -1.$$
>
> This means
>
> $$|\ddot{\alpha}(0) \cdot \alpha(0)| \geq 1.$$
>
> Since $k = |\ddot{\alpha}(0)|$ and $|\alpha(0)| = 3$, the Cauchy-Schwarz inequality implies
>
> $$3k = |\alpha(0)||\ddot{\alpha}(0)| \geq |\ddot{\alpha}(0) \cdot \alpha(0)| \geq 1.$$
>
> Dividing by 3 now gives the result.

2. (25 points) (3-3.2) A line passing through $(0,0,a) \in \mathbb{R}^3$ is parameterized by

$$\gamma(t) = (0,0,a) + t(\cos a, \sin a, 0).$$

Find a parameterization for the surface which is the union of all such lines.

> **Solution:**
>
> $$X(u,v) = (u\cos v, u\sin v, v).$$

3. (25 points) (3-2.8, Proposition 2, page 167) Let

$$\mathcal{S}_r = \{(x, y, x^2 + y^2) : x^2 + y^2 < r^2\}.$$

   1. Find an expression for the area of the image of the Gauss map of $\mathcal{S}_r$.

2. Find an expression for the area of $\mathcal{S}_r$.
3. Denote the area of $\mathcal{S}_r$ by $A(r)$ and the area of the Gauss image by $\mathcal{G}(r)$. Compute

$$\lim_{r\to 0}\frac{\mathcal{G}(r)}{A(r)}.$$

Solution:

$$A(r) = 2\pi\int_0^r \sqrt{1+4t^2}\,tdt = \frac{\pi}{6}[(1+4r^2)^{3/2}-1].$$

$\mathcal{G}(r)$ is the area of a spherical cap on the unit sphere above the ball of radius $2r/\sqrt{1+4r^2}$. That is,

$$\mathcal{G}(r) = 2\pi\int_0^{\frac{2r}{\sqrt{1+4r^2}}}\sqrt{1+\frac{t^2}{1-t^2}}\,tdt = 2\pi\left[1-\frac{1}{\sqrt{1+4r^2}}\right].$$

Thus,

$$\frac{\mathcal{G}(r)}{A(r)} = 12\frac{1-(1+4r^2)^{-1/2}}{(1+4r^2)^{3/2}-1}.$$

One application of L'Hopital's rule gives that the desired limit of this expression is 4, which is the Gauss curvature of the paraboloid at the origin.

4. (25 points) (4-2.4,12) Find an isometry from

$$B_1 = \{(x,y,0) : x^2+y^2<1\}$$

to a surface $\mathcal{S}$ of constant mean curvature $\pi/2$. Specify clearly the surface $\mathcal{S}$ and the isometry.

Solution: Since B_1 is flat, $\mathcal{S}$ must have Gauss curvature $K\equiv 0$. A cylinder is an obvious choice. In order to get $H=\pi/2$, we take

$$\mathcal{S} = \{(x,y,z) : x^2+y^2 = 1/\pi^2\}.$$

Let $\phi : B_1 \to \mathcal{S}$ by

$$\phi(x,y,0) = \left(\frac{\cos(\pi x)}{\pi}, \frac{\sin(\pi x)}{\pi}, y\right).$$

(ϕ wraps the disk around the cylinder.) Observe that ϕ is one-to-one since the curcumference of $\mathcal{S}$ is the same as the diamter of B_1. Also, we have

$$\phi_x = (-\sin(\pi x), \cos(\pi x), 0) \quad \text{and} \quad \phi_y = (0,0,1).$$

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Thus, $|\phi_x| = 1$, $\phi_x \cdot \phi_y = 0$, and $|\phi_y| = 1$. Given $v = (a, b) \in T_pB_1$ and $w = (c, d) \in T_p(B_1)$, we have

$$d\phi(v) = \phi_x a + \phi_y b \quad \text{and} \quad d\phi(w) = \phi_x c + \phi_y d.$$

Thus,

$$\begin{aligned}\langle d\phi(v), d\phi(w)\rangle &= \langle \phi_x a + \phi_y b, \phi_x c + \phi_y d\rangle \\ &= ac + bd \\ &= \langle v, w\rangle.\end{aligned}$$

Thus, ϕ is an isometry.