Remarks on curvature in Riemannian geometry

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The objective here is to relate various aspects of Riemann's notion of curvature for manifolds to more elementary concepts of curvature for curves and surfaces. This development follows (or attempts to follow) at least in spirit the introductions to intrinsic derivatives discussed earlier.

We also discuss/review the curvature of a regularly parameterized surface in $\mathbb{R}^3$ in detail. As far as I know there are two main sources of intuition for Riemann's basic ideas about manifolds. Those are experience with submanifolds of Euclidean space and consideration of changes of coordinates for flat Euclidean space $\mathbb{R}^n$. The example of a surface $\mathcal{S} \subset \mathbb{R}^3$ falls into the first category.

More generally, it is from these sources that one can come to the conclusion that the intrinsic derivative of a vector field $w \in \mathfrak{X}(M)$ defined on a manifold M should exist (and is a natural object with certain properties). From this perspective it is not unreasonable to at least believe a Riemannian manifold M is somehow intrinsically "curved." In particular, it is hoped that the consideration of specific familiar examples, like a two-dimensional spherical cap intrinsically associated with a metric assignment on a domain in $\mathbb{R}^2$, is a reasonable source for such a belief.

1 Review of intrinsic derivatives

Recall that the concept of an intrinsic derivative of a real valued function $f : M \to \mathbb{R}$ defined on a manifold M rested on the rate of change of a real valued function with respect to the distance of displacement in the manifold determined by the Riemannian metric tensor. This rate of change at a point $P \in M$ was denoted by

$$D_u f(P)$$

where $u \in T_P M$ was a unit filament vector. This rate of change was expressed as a difference quotient using the convenient fact that values of f at different points on M may be freely subtracted from one another.

The algebraic extension of the values of directional derivatives to other vectors $z \in T_P M$ were denoted by

$$df_P(z)$$

where $df_P : T_P M \to \mathbb{R}$ is a point differential map. Finally, the algebraic extension to the point differential was considered at each point on the manifold to obtain a spatially extended, global, or general, differential

$$df : \mathfrak{X}(M) \to cC^\infty(M)$$

which was $cC^\infty(M)$ (module) linear. This object was an example of a $(0, 1)$ tensor, and was metrically equivalent to a certain vector field Df in the sense that

$$\mu(Df, z) = df(z)$$

where $\mu : \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M)$ was the $(0, 2)$ Riemannian metric tensor.

The situation was more problematic when we wanted to consider the derivative of a vector field $w \in \mathfrak{X}(M)$ owing to the fact that there is no natural way to determine in general the difference of the values w_P and w_Q where P and Q are distinct points in M. Equipped with the belief that a directional derivative

$$\nabla_u w(P)$$

should exist for $u \in \mathbb{S}^1_P \subset T_P M$, and be a naturally defined geometric object, we listed various properties such a derivative might have—and perhaps should have. Among those were the belief that $\nabla_u w(P)$ should be a vector in $T_P M$ and have an algebraic extension $dw_P : T_P M \to T_P M$. This extension is, of course, a point differential for the vector field w at P, and it is natural to also imagine there should be an extension $dw : \mathfrak{X}(M) \to \mathfrak{X}(M)$ to a global differential on the manifold M. With the addition/discovery of two key properties, **metric compatibility** according to which the differential of the real valued function $\mu(w_1, w_2)$ is related to the vector field differentials dw_1 and dw_2 by

$$d\mu(w_1, w_2) = \mu(dw_1, w_2) + \mu(w_1, dw_2) \qquad \text{for} \qquad w_1, w_2 \in \mathfrak{X}(M)$$

and **symmetry** according to which

$$dv_j(v_i) = dv_i(v_j)$$

for $\{v_1, v_2, \ldots, v_n\}$ local vector fields determined by a chart $(U, \mathbf{p})$ so that $v_j = d\mathbf{p}(\mathbf{e}_j)$ for $j = 1, 2, \ldots, n$, we were able to identify $\nabla_u w$ with a formula in terms of known quantities associated with a chart. Specifically, the local vector fields

$$dv_i(v_j) = \sum_{k=1}^{n} \Gamma^k_{ij}\ v_k$$

and the associated coefficients Γ^k_{ij} called Christoffel symbols played a central role in the formula.

There still remains some work to do in verifying this local formula can be used

(a) to define a unique vector $\nabla_u w(P)$ for each $u \in \mathbb{S}^1_P$,

(b) to define by algebraic extension a point differential map $dw_P : T_P M \to T_P M$, and

(c) to define a global differential map $dw : \mathfrak{X}(M) \to \mathfrak{X}(M)$

all having all the properties desired/needed desired to be associated to the intrinsic derivative of a vector field. As we proceed, we will assume this development has been accomplished and is at our disposal both as a technical/quantitative device for differentiating vector fields and as a model for the development of the next basic concept of Riemannian geometry, namely curvature.

Our discussion of intrinsic derivatives illustrates a general pattern of starting with a heuristic understanding (or belief in) some geometric quantity or object, followed by some specific determination of a quantitative value for that quantity, perhaps through intermediary assumptions about certain algebraic and spatial extensions and various properties. The approach to curvature below attempts to follow a similar pattern.

2 Two dimensional surfaces in $\mathbb{R}^3$

The key observation about two dimensional surface in $\mathbb{R}^3$ is associated with the French mathematician Jean Baptiste Marie Charles Meusnier de la Place better known today simply as Meusnier (pronounced Moon-yea). Meusnier's theorem says roughly that the curvature of a surface $\mathcal{S} \subset \mathbb{R}^3$ is "about" the curvature of curves (passing through a point) in $\mathcal{S}$. Specifically, consider a path $\gamma : I \to \mathcal{S}$ parameterized by arclength so that $\dot\gamma(s) \in \mathbb{S}^1_{\gamma(s)}$ for $s \in I$ and with $\gamma(s_0) = P \in \mathcal{S}$. Here we are taking

$$\mathbb{S}^1_P = \{\mathbf{u} \in T_P\mathcal{S} : |\mathbf{u}| = 1\}$$

the collection of traditional vectors in $\mathbb{R}^3$ comprising the unit circle in $T_P\mathcal{S}$, also taken to consist of traditional vectors in $\mathbb{R}^3$, though $M = \mathcal{S}$ can also be considered as a Riemannian manifold with the usual filament vectors determiined by paths and

$$\mu_P([\alpha_1], [\alpha_2]) = \langle \alpha'(t_1), \alpha'(t_2) \rangle_{\mathbb{R}^3}. \tag{1}$$

Exercise 1 Assume $X = \mathbf{p} : U \to \mathcal{S}$ is a parameterization of the set $\mathbf{p}(U) \subset \mathcal{S}$ defined on the open set $U \subset \mathbb{R}^2$ and is also a chart function for the Riemannian manifold $M = \mathcal{S}$. Find the metric coefficients $g_{ij} : U \to \mathbb{R}$ for $i, j = 1, 2$ so that

$$\mu_X([\alpha_1], [\alpha_2]) = \langle g \; d\xi_X(\alpha_1'(t_1)), d\xi_X(\alpha_2'(t_2)) \rangle_{\mathbb{R}^2}$$

agrees with the assignment (1) associated with the inner product induced from $\mathbb{R}^3$ where $g = (g_{ij}(\mathbf{x}))$ denotes the matrix of metric coefficients at $\mathbf{x} = \xi(P)$.

There are many such paths γ as illustrated in Figure 1.

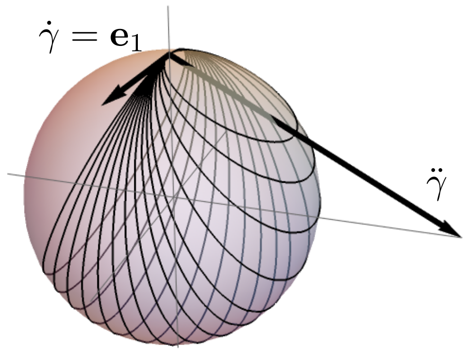

Figure 1: Many paths on a spherical surface sharing the same tangent vector $\dot{\gamma} \in \mathbb{S}^1_P$ at the north pole P. The curvature vector $\ddot{\gamma}$ of the smallest circle through P is indicated.

The **curvature vector** of such a path $\gamma : I \to \mathcal{S}$ at P is given by

$$\ddot{\gamma}(s_0) = \frac{d^2\gamma}{ds^2}(s_0)$$

the second derivative of γ with respect to arclength. Meusnier asserts that the quantity

$$k_\gamma = \langle \ddot{\gamma}, N \rangle_{\mathbb{R}^3},$$

where $N = N_P$ is a specified unit normal to the surface $\mathcal{S}$ at P, is independent of the particular path γ and depends only on the direction determined by the unit vector $\dot{\gamma}(s_0)$. Accordingly, the value k_γ is called the **normal curvature** of $\mathcal{S}$ at P in the direction $\dot{\gamma} \in \mathbb{S}^1_P$ and gives rise to a function

$$II : \mathbb{S}^1_P \to \mathbb{R} \qquad \text{by} \qquad II(\dot{\gamma}) = \langle \ddot{\gamma}, N \rangle_{\mathbb{R}^3}.$$

The proof of Meusnier's theorem is obtained by considering the quantity $\langle \dot{\gamma}, N \rangle_{\mathbb{R}^3}$ as a function of s along the entire path which happens to take the constant value zero. Differentiating according to the usual metric compatibility of the Euclidean inner product one finds

$$\langle \ddot{\gamma}, N \rangle_{\mathbb{R}^3} = -\langle \dot{\gamma}, \dot{N} \rangle_{\mathbb{R}^3}, \tag{2}$$

where of course, $N : I \to \mathbb{R}^3$ is a unit normal field to $\mathcal{S}$ extending $N = N_P$ along the path determined by γ. Since

$$\dot{N} = \frac{dN}{ds}(s_0)$$

is independent of the particular path γ but depends only on $\dot{\gamma}(s_0)$. To see this more clearly and also discern something else interesting from the Meusnier relation (2) consider an extension $\bar{N} : W \to \mathbb{R}^3$ to an open set $W \subset \mathbb{R}^3$ with $P \in W$ giving also a unit normal field on $\mathcal{S}$. Then

$$\dot{N} = D\bar{N}(\gamma)\,\dot{\gamma}.$$

Not only does this expression make it clear that $\dot{N} = \dot{N}(s_0)$ depends only on the direction $\dot{\gamma}(s_0)$ since $D\bar{N}(P)$ is entirely independent of γ, but the function

$$S : \mathbb{S}^1_P \to T_P\mathcal{S} \qquad \text{by} \qquad S(\dot{\gamma}) = -\dot{N}$$

extends (algebraically) to a linear function

$$S \in \mathcal{L}(T_P\mathcal{S} \to T_P\mathcal{S}) \qquad \text{with} \qquad S(\mathbf{z}) = -D\bar{N}(P)\,\mathbf{z}.$$

Exercise 2 Show there exists a vector field $\bar{N} : V \to \mathbb{R}^3$ giving an extension of a specified normal field N to $\mathcal{S}$ defined in an open set $\mathbf{p}(U) \subset \mathcal{S}$ containing P.

Exercise 3 Show that for $\mathbf{z} \in T_P\mathcal{S}$ the vector

$$S(\mathbf{z}) = -D\bar{N}(P)\,\mathbf{z}$$

satisfies $S(\mathbf{z}) \in T_P\mathcal{S}$.

Exercise 4 Show the curvature vector $\ddot{\gamma} = \ddot{\gamma}(s_0)$ at $P \in \mathcal{S}$ determined by a path $\gamma : I \to \mathcal{S}$ parameterized by arclength on the surface $\mathcal{S}$ satisfies

$$\langle \ddot{\gamma}, \dot{\gamma} \rangle_{\mathbb{R}^3} = 0$$

so that the curvature vector $\ddot{\gamma}$, when nonvanishing, determines a unique direction

$$\frac{\ddot{\gamma}}{|\ddot{\gamma}|} \in \mathbb{S}^2_P \subset \mathbb{R}^3$$

normal to the path determined by γ called the **principal normal** to the path/curve. The principal normal vector satisfying

$$\ddot{\gamma} = |\vec{k}|\,\frac{\ddot{\gamma}}{|\ddot{\gamma}|},$$

when considered in the theory of curves, is also often denoted by N so that the curvature vector $\vec{k}$ satisfies $\vec{k} = \ddot{\gamma} = |\vec{k}|\,N$, but in the present context one should be careful to realize this is not always a normal to $\mathcal{S}$.

We have now seen that the fundamental geometric quantity $II(\dot{\gamma})$ called the normal curvature of $\mathcal{S}$ in the direction $\dot{\gamma}$ satisfies

$$II(\dot{\gamma}) = \langle \dot{\gamma}, S(\dot{\gamma}) \rangle_{\mathbb{R}^3} = \langle \dot{\gamma}, -d\bar{N}\,\dot{\gamma} \rangle_{T_P\mathcal{S}}.$$

Consequently, there is an (algebraic) extension $Q : T_P\mathcal{S} \to \mathbb{R}$ of $II : \mathbb{S}^1_P \to \mathbb{R}$ which is a **quadratic form** satisfying

$$Q(c\mathbf{z}) = c^2 Q(\mathbf{z}) \qquad \text{for} \qquad \mathbf{z} \in T_P\mathcal{S},\ c \in \mathbb{R}.$$

The function $II : \mathbb{S}^1_P \to \mathbb{R}$, or sometimes more generally the algebraically extended $Q : T_P\mathcal{S} \to \mathbb{R}$, is called the **second fundamental form** of the surface $\mathcal{S}$. As with all quadratic forms there is a bilinear form

$$B : T_P\mathcal{S} \times T_P\mathcal{S} \to \mathbb{R} \qquad \text{by} \qquad B(\mathbf{z}_1, \mathbf{z}_2) = \langle \mathbf{z}_1, S(\mathbf{z}_2) \rangle_{\mathbb{R}^3} = \langle \mathbf{z}_1, S(\mathbf{z}_2) \rangle_{T_P\mathcal{S}}$$

satisfying

$$Q(\mathbf{z}) = B(\mathbf{z}, \mathbf{z}).$$

The function $S \in \mathcal{L}(T_P\mathcal{S} \to T_P\mathcal{S})$ with $S(\dot\gamma) = -\dot N$ is called the **shape operator** of the surface.

It is quite reasonable to say at this point that the "curvature" of $\mathcal{S}$ is (completely) captured by the values of the normal curvatures or by the values of the second fundamental form, or any of the associated extensions and/or constructions like the shape operator from which one can recover the values of the normal curvatures of Meusnier. Before I proceed further to analyze and consolidate what can be said about the normal curvatures, I wish to augment the general discussion of the curvature of $\mathcal{S}$ given above in two additional and roughly equivalent ways. First I wish to consider carefully the curvature vectors $\vec{k} = \ddot\gamma$ above and determine special representative curves for the direction $\dot\gamma$ at each point P. Second I wish to consider the values of the shape operator on these special representative curves. The special representative curves turn out to be **geodesics**, so roughly speaking, I want to incorporate the role played by geodesics in relation to normal curvatures or more properly introduce geodesics into the discussion.

I return to consideration of the family of space curves parameterized by arclength, lying in a surface $\mathcal{S} \subset \mathbb{R}^3$, and passing through a particular point $P \in \mathcal{S}$ with a common tangent $\dot\gamma(s_0)$. While the condition $|\dot\gamma| = 1$ is a condition having to do with specific parameterization of say $\gamma \in \mathfrak{I}^\infty(\mathcal{S})$, the curvature vector $\vec{k} = \ddot\gamma \in \mathbb{R}^3$ and specifically $\ddot\gamma(s_0)$ is a fundamentally geometric quantity determined by the curve independent of parameterization. This vector, furthermore, admits a decomposition

$$\ddot\gamma = k_\gamma\ N + k_g\ \dot\gamma^\perp$$

where as discussed above the normal curvature $k_\gamma = \langle \ddot\gamma, N\rangle_{\mathbb{R}^3}$ is independent of the particular path in the class under consideration. The other component $k_g\ \dot\gamma^\perp = \ddot\gamma - k_\gamma\ N$ depends on the path, is orthogonal to the path, and is tangent to $\mathcal{S}$. This tangent component $k_g\ \dot\gamma^\perp$ of the curvature vector $\ddot\gamma$ is called the **geodesic curvature vector**. Among the paths under consideration at P, there is a locally unique one for which the condition

$$k_g = 0$$

holds identically, and this is called the **geodesic** passing through P in the direction $\dot\gamma = \dot\gamma(s_0)$. Taking this particular path in the forward direction for some positive length ϵ (small enough) we can express some open set $\mathbf{p}(G) \subset \mathcal{S}$ as

$$\mathbf{p}(G) = \{P\} \cup \{\gamma(s) : s_0 < s < s_0 + \epsilon,\ \dot\gamma(s_0) \in \mathbb{S}^1_P\} \tag{3}$$

where G is an open set in $U \subset \mathbb{R}^2$ with $\mathbf{x} = \mathbf{p}^{-1}(P) \in G$. The expression on the right in (3) gives the **geodesic decomposition** of $\mathcal{S}$ near P or the geodesic star at P as illustrated for a spherical surface in Figure 2.

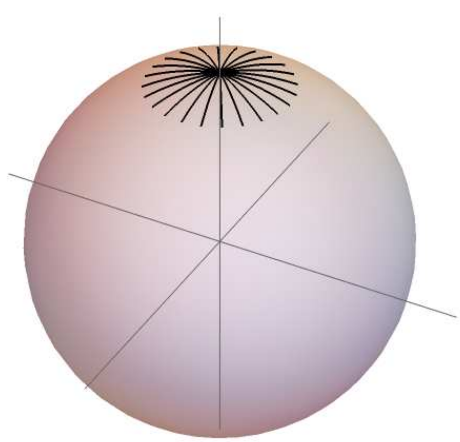

Figure 2: Geodesic star at the north pole on the sphere.

3 Summary lessons

Hopefully one comes away from the discussion above believing **curvature is about submanifolds**. Specifically, the curvature of a higher dimensional object (in this case a two dimensional surface $\mathcal{S}$) is described in terms of one dimensional submanifolds (paths) passing through P. In addition, one can think of the restriction $|\dot\gamma| = 1$ as analogous to the restriction $u \in \mathbb{S}^1_P$ associated with the directional derivative(s) $D_u f$ or $\nabla_u w$. Notice the parallels:

$$II(\dot\gamma) = \langle \ddot\gamma, N\rangle_{\mathbb{R}^3} \quad \text{extends to} \quad Q : T_P\mathcal{S} \to \mathbb{R};$$

$$D_u f(P) \qquad \text{extends to} \quad df_P : T_P M \to \mathbb{R}.$$

$$S : T_P\mathcal{S} \to T_P\mathcal{S} \qquad \text{extends spatially to} \qquad S : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S});$$

$$dw_P : T_P M \to T_P M \quad \text{extends spatially to} \quad dw : \mathfrak{X}(M) \to \mathfrak{X}(M).$$

An important feature in the discussion above is that if γ is a geodesic with $|\dot\gamma| = 1$, i.e., a unit speed geodesic, then

$$\ddot\gamma = k_\gamma \, N.$$

Here an extrinsic curvature is naturally attached to an extrinsic normal direction. On the other hand, starting with a geometrically meaningful quantity $II(\dot\gamma) = k_\gamma$ (Meusnier's normal curvature), we obtain a linear operator $S : T_P\mathcal{S} \to T_P\mathcal{S}$ (the shape operator) which also extends spatially to a function $S : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$ which is $C^\infty(\mathcal{S})$ (module) linear. Observe that here for a unit speed geodesic one has

$$S(\dot\gamma) = -\dot N = k_\gamma \, \dot\gamma$$

because

$$k_\gamma = \langle \ddot\gamma, N\rangle_{\mathbb{R}^3} = \langle \dot\gamma, -\dot N\rangle_{T_P\mathcal{S}}.$$

Thus, S is giving a tangential expression for the normal curvature.

Finally, as something of an aside we mention/recall that there is an alternative form of the shape operator obtained by taking the inner product with an additional argument vector:

$$S^* : \mathfrak{X}(\mathcal{S}) \times \mathfrak{X}(\mathcal{S}) \to C^\infty(\mathcal{S}) \qquad \text{by} \qquad S^*(\mathbf{z}_1, \mathbf{z}_2) = \langle S(\mathbf{z}_1), \mathbf{z}_2\rangle.$$

The function S^* is (module) linear over $C^\infty(\mathcal{S})$, and is (therefore) a tensor, in this case a $(0,2)$ tensor.

4 Intrinsic curvature versus extrinsic curvature

Gauss pointed out that the second fundamental form $II : \mathbb{S}^1_P \to \mathbb{R}$ and/or the shape operator $S : T_P\mathcal{S} \to T_P\mathcal{S}$ contains **both**

(i) information about the way $\mathcal{S}$ is curved in $\mathbb{R}^3$, and

(ii) other information about the way $\mathcal{S}$ is curved which can be determined strictly using the metric tensor, that is for example the local metric coefficients $g_{11} = E$, $g_{12} = F$, and $g_{22} = G$ intrinsic to $\mathcal{S}$.

This is essentially the subject of Gauss' Theorema Egregium. The intrinsic information on the curvature of a surface $\mathcal{S}$ at a point $P \in \mathcal{S}$ reduces to one number

$$K = \det(S).$$

This number, as might be expected, is independent of the choice of normal N used to define the shape operator S. This might be expected because the choice of N is fundamentally an extrinsic geometric construction. Going back to the normal curvatures, we can say a little more. If one looks at the values k_γ as $\dot\gamma$ varies in $\mathbb{S}^1_P$, one always sees a maximum value $k_{\max}$ and a minimum value $k_{\min}$. These are always taken at tangent directions which are orthogonal. This interesting fact is the result of the fact that the shape operator is self-adjoint with respect to the inner product:

$$\langle S(\mathbf{z}_2), \mathbf{z}_1\rangle = \langle S(\mathbf{z}_1), \mathbf{z}_2\rangle.$$

or equivalently that the bilinear form $B : T_P\mathcal{S} \to T_P\mathcal{S} \to \mathbb{R}$ associated with the algebraic extension Q of II is symmetric. As a result of these algebraic properties one finds

$$K = k_{\min}\ k_{\max} = II(\dot\gamma_{\min})\ II(\dot\gamma_{\max})$$

with $\langle \dot\gamma_{\min}, \dot\gamma_{\max}\rangle = 0$. The quantity K is called the **Gauss curvature** of $\mathcal{S}$ at P. The proof of Gauss' theorem is essentially the following: The second partial derivatives of the parameterization vector $X : U \to \mathcal{S} \subset \mathbb{R}^3$ satisfy

$$\begin{aligned} X_{x_1x_1} &= \Gamma^1_{11}\ X_{x_1} + \Gamma^2_{11}\ X_{x_2} + e\ N \\ X_{x_1x_2} &= \Gamma^1_{12}\ X_{x_1} + \Gamma^2_{12}\ X_{x_2} + f\ N \\ X_{x_2x_2} &= \Gamma^1_{22}\ X_{x_1} + \Gamma^2_{22}\ X_{x_2} + g\ N \end{aligned}$$

where the classical/traditional Christoffel symbols

$$\Gamma^k_{ij} : U \to \mathbb{R}$$

are considered (naturally) as functions with domain in a chart $U \subset \mathbb{R}^2$ and can be expressed only in terms of the metric coefficients (of the first fundamental form). Thus, the Christoffel symbols are intrinsic, while the coefficients $\ell_{11} = e$, $\ell_{12} = f$, and $\ell_{22} = g$ of the second fundamental form are extrinsic. Gauss showed, however, that

$$K = \det(S) = k_{\min}\ k_{\max} = \frac{1}{g_{11}}\left(\frac{\partial\Gamma^2_{11}}{\partial x_2} - \frac{\partial\Gamma^2_{12}}{\partial x_1} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 + \Gamma^2_{11}\ \Gamma^2_{22} + \Gamma^1_{11}\ \Gamma^2_{12}\right). \tag{4}$$

It's perhaps a bit of an ugly calculation, but of course a very pretty theorem. This, it turns out, is the main quantity of interest for curvature in Riemannian geometry. Before discussing how and why that is the case, I will present something of an aside.

Any of the value of the normal curvature k_γ is an extrinsic curvature. There are essentially a (half) circle's worth of these values and they contain $k_{\min}$ and $k_{\max}$. These values depend on the choice of the (local) normal N in particular. If the other normal $-N$ to $\mathcal{S}$ is taken at $P \in \mathcal{S}$, then the normal curvatures in the directions $\dot\gamma_{\min}$ and $\dot\gamma_{\max}$ change signs. As a result, the minimum becomes a maximum and the maximum becomes a minimum. The product K however does not change, and in fact according to (4) the value of K may be computed without reference to anything except distances and angles measured using paths $\alpha \in \mathfrak{P}^\infty(\mathcal{S})$.

One particular single number associated with the extrinsic curvature of a surface or the way the surface is curving in the ambient three-dimensional space $\mathbb{R}^3$ is given by the average value

$$H = \frac{1}{2}\,\mathrm{Trace}(S) = \frac{k_{\min} + k_{\max}}{2}.$$

This value does depend on the choice of N and changes sign when N changes. This is called the **mean curvature** of the surface $\mathcal{S}$ at the point $P \in \mathcal{S}$. One quite interesting thing about mean curvature is that one need not use the particular orthogonal directions $\dot\gamma_{\min}$ and $\dot\gamma_{\max}$ to compute it: Given any orthonormal directions $\mathbf{u}_1 = \dot\gamma$ and $\mathbf{u}_2 = \dot\gamma^\perp$ in $\mathbb{S}^1_P \subset T_P\mathcal{S}$, one has

$$H = \frac{II(\dot\gamma) + II(\dot\gamma^\perp)}{2} = \frac{II(\mathbf{u}_1) + II(\mathbf{u}_2)}{2}. \tag{5}$$

Exercise 5 Write down a one parameter family of explicit isometric flexings of the flat plane $\{(x, y, 0) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ into cylindrical surfaces. Find the Gauss curvature and mean curvature of each of these surfaces.

Exercise 6 Show in general that

$$K = \frac{eg - f^2}{EG - F^2}$$

and

$$2H = \frac{Eg - 2fF + eG}{EG - F^2}.$$

5 Flat surfaces in $\mathbb{R}^3$

There are other flexings of the plane in $\mathbb{R}^3$, and all such surfaces have been classified...

6 Curvature of manifolds

6.1 One dimensional manifolds

There is no intrinsic notion of curvature for one dimensional manifolds. This is in contrast to the curvature of curves in the plane which is nicely given in terms of the curvature vector or curves in $\mathbb{R}^3$ whose curvature is partially described by the curvature vector discussed above in the context of surfaces in $\mathbb{R}^3$. These are extrinsic notions of the curvature of a one dimensional manifold as a submanifold of a higher dimensional manifold or ambient space. Note also that what we are saying here is not that one dimensional manifolds have zero curvature, but rather that there are not enough dimensions for any notion of curvature whatsoever. Specifically, if you have a vector field w in a one dimensional manifold and a unit field u, which you can always have in a one-dimensional manifold, then the vector field $\nabla_u w$ can only tell you about how the length of w is changing. Curvature is fundamentally about how the direction of vector fields change, and intrinsic curvature is about how the directions of vector fields, in a certain sense, **have to** or or **forced** to change. There is no notion of the change of direction[1] of a vector field in a one dimensional manifold.

6.2 Two dimensional manifolds

At each point P in a two dimensional manifold M there is a unique geodesic star or local geodesic decomposition similar to the one described above for surfaces. We have not developed adequately the properties of such a decomposition, but we are pretty close. The main tool is the existence and uniqueness theorem for ordinary differential equations.

Given the geodesic star at $P \in M$ in a two dimensional manifold, the Gauss formula (4) may be adapted in the form

$$\begin{aligned} K(P) = \frac{1}{g_{11} \circ \xi(P)} \big[& d(\Gamma^2_{11})_P(v_2) - d(\Gamma^2_{12})_P(v_1) \\ & - \Gamma^1_{12}(P)\,\Gamma^2_{11}(P) - (\Gamma^2_{12}(P))^2 + \Gamma^2_{11}(P)\,\Gamma^2_{22}(P) + \Gamma^1_{11}(P)\,\Gamma^2_{12}(P) \big] \end{aligned} \tag{6}$$

where the Christoffel symbols are locally defined real valued functions on M with respect to a chart $(U, \mathbf{p})$ with $P \in \mathbf{p}(U)$ satisfying

$$dv_i(v_j) = \sum_{k=1}^{2} \Gamma^k_{ij} v_k \qquad \text{for} \qquad i, j = 1, 2, \ldots, n$$

[1] Of course, one can say a vector field can change in direction from "forward" to "backward" in a one dimensional manifold, but this is fundamentally just a matter of the change of length. In particular, the length must vanish for such a change of direction. That kind of change in direction is not what curvature is about.

and $\{v_1, v_2\}$ the coordinate basis induced by the chart $(U, \mathbf{p})$ as usual.

One needs to show (6) is independent of the particular chart chosen. Then this gives a notion of intrinsic curvature for M^2. In particular, a two dimensional manifold M is said to be **negatively curved** at P if $K(P) < 0$, **positively curved** at P if $K(P) > 0$, and **flat** at P if $K(P) = 0$. In general, the value of K can change from point to point and can be positive and negative on a single two dimensional manifold.

Exercise 7 Show that a standard (two dimensional) torus in $\mathbb{R}^3$ has Gauss curvature taking all values on some interval $[K_{\min}, K_{\max}]$ where $K_{\min} < 0 < K_{\max}$ as a surface in $\mathbb{R}^3$. For exmaple, you can take the stereographic projection of the Clifford torus

$$\mathcal{C} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = 1 = x_3^2 + x_4^2\} \subset \mathbb{R}^4.$$

Show the same surface takes the same values for the curvature as a Riemannian manifold. Define a manifold M homeomorphic to $\mathcal{C}$ and give it a (flat) Riemannian structure so that $K \equiv 0$ on M.

The two dimensional manifolds of constant Gauss curvature have been classified. In particular, these are determined locally in the sense that there is a neighborhood $\mathbf{p}(U) \subset M$ in every such manifold upon which the Riemannian metric tensor μ is determined uniquely by the constant $K \in \mathbb{R}$. Every such manifold admits a real analytic atlas, and each such neighborhood is embeddable in $\mathbb{R}^3$.

6.3 Three dimensional manifolds

For higher dimensional manifolds the intrinsic curvature is understood/determined/defined according to the principle announced above that the curvture is, first of all, about submanifolds. Second, in order to discuss the basic object of geometric meaning, it is convenient to consider certain orthonormal bases at a point $P \in M$. Some of our initial considerations will be general. There is always a decomposition for some neighborhood $\mathbf{p}(U) \subset M$ as a geodesic star. In particular, for the appropriate U, there is some $\epsilon > 0$ for which

$$\mathbf{p}(U) = \{P\} \cup \{\gamma(s) : s_0 < s < s_0 + \epsilon, \ [\dot{\gamma}(s_0)] \in \mathbb{S}^{n-1}(P)\}$$

with all the (geodesic) filaments disjoint.[2] See Figure 2 above. Given a **pair** of orthonormal vectors u_1 and u_2 in $\mathbb{S}_P^{n-1} \subset T_P M$, the set

$$\mathcal{S}(u_1, u_2) = \{P\} \cup \{\gamma(s) : s_0 < s < s_0 + \epsilon, \ [\dot{\gamma}(s_0)] \in \operatorname{span}\{u_1, u_2\}\}$$

is an open two dimensional submanifold of M. Note: These submanifolds cannot be expected to always be disjoint. No pair

$$(\mathcal{S}(u_1, u_2)\backslash\{P\}, \mathcal{S}(\tilde{u}_1, \tilde{u}_2)\backslash\{P\})$$

will be disjoint when $n = 3$. However, the curvature of M^3 is captured by the collection of numbers

$$K(u_1, u_2)$$

of Gauss curvatures of two dimensional (totally geodesic) submanifolds $\mathcal{S} \subset M$ tangent to $\operatorname{span}\{u_1, u_2\}$ at P where $\{u_1, u_2\}$ ranges over all pairs of orthonormal bases in $T_P M$.

The structure upon which this notion of the curvature of M is based should be, at least in some sense, naturally comparable to the notion of curvature for surfaces $\mathcal{S}$ in $\mathbb{R}^3$ discussed above. Here are the parallels:

M	decomposes locally at $P \in M$	into totally geodesic two dimensional submanifolds $\mathcal{S}(u_1, u_2)$;
$\mathcal{S} \subset \mathbb{R}^3$	decomposes locally at $P \in \mathcal{S}$	into geodesics γ.

[2]Again, this is an assertion we have not fully justified, but we have gone a good deal in this direction by writing down the ordinary differential equation for geodesics.

$$\text{The (intrinsic) curvature of } M \text{ at } P \quad \text{is given by} \quad \{K(u_1,u_2) : \mu_P(u_i,u_j) = \delta_{ij}\};$$

$$\text{The (extrinsic) curvature of } \mathcal{S} \text{ at } P \quad \text{is given by} \quad \{\ddot{\gamma}(s_0) = k_\gamma : \dot{\gamma}(s_0) \in \mathbb{S}^1_P\}.$$

There are also distinctions to be noted. The local decomposition of $\mathcal{S} \subset \mathbb{R}^3$ is a partition, meaning the geodesic filaments are disjoint except at P and thus the decomposition of the curvature of the surface into normal curvatures can be viewed as a disjoint curvature decomposition. The geodesic filaments are similarly individually disjoint in higher dimensions as well, but the collections $\mathcal{S}(u_1,u_2)$ comprising two dimensional submanifolds are not disjoint in general, so the intrinsic curvature decomposition is not a partition with respect to the sets of which the (Gauss) curvature is measured. For this reason it may be expected that the totally geodesic two dimensional submanifolds $\mathcal{S}(u_1,u_2)$ may have curvatures that display a more complicated structure among themselves. Riemann's idea of curvature is a fundamentally new and different kind of idea in this sense.

The basic geometric object through which the intrinsic curvature of a Riemannian manifold[3] is understood is the Gauss curvature $K(u_1,u_2)$ of totally geodesic two dimensional submanifolds determined by orthonormal bases $\{u_1,u_2\} \subset \mathbb{S}^{n-1}_P$. Each such curvature $K(u_1,u_2)$ is called the **sectional curvature** determined by $\{u_1,u_2\}$. This was the idea of Riemann and presumably his adaption of Gauss' work on the curvature of surfaces.

The Gauss curvature and the mean curvature of a surfaces $\mathcal{S} \subset \mathbb{R}^3$ may be viewed as algebraic expressions involving the (basic geometric) normal curvatures. Furthermore, as we have seen above, the basic geometric normal curvature $II(\dot{\gamma})$ considered as a function on $\mathbb{S}^1_P \subset T_P\mathcal{S}$ leads to other algebraic extensions with various properties which one can say loosely[4] are "linearity properties."

It is to be expected that there is an extension of the sectional curvature

$$K : \{(u_1,u_2) \in \mathbb{S}^{n-1}_P \times \mathbb{S}^{n-1}_P \to \mathbb{R} : \mu_P(u_i,u_j) = \delta_{ij},\ j = 1,2\} \to \mathbb{R}$$

with some kind of linearity properties. Also, like the normal curvature $II(\dot{\gamma}) = k_\gamma$ has associated vector expressions

$$\ddot{\gamma} = k_\gamma\, N \qquad \text{and} \qquad S(\dot{\gamma}) = -\dot{N} = k_\gamma\, \dot{\gamma}, \tag{7}$$

and it is natural to imagine there is a vector expression

$$R_0(u_1,u_2) = K(u_1,u_2)\, u \tag{8}$$

for some $u \in T_PM$, though Riemann did now write about any of these "algebraic" developments. As a first observation in making such a vector identification, specifically in thinking about how to identify an appropriate vector u for which to define some Riemannian curvature vector R_0 as indicated in (8) we can perhaps note that, while the basic idea may be inspired by the expressions in (7) this is one instance where fundamental differences should be taken into account.

In looking at the formula $\ddot{\gamma} = k_\gamma\, N$ one might be initially inclined to take the vector u in (8) to be a vector orthogonal to u_1 and u_2, espectially if M is three dimensional. The point to percieve however is that $\ddot{\gamma}$ and the associated normal curvature k_γ is fundamentally an extrinsic curvature, while the sectional curvature $K(u_1,u_2)$ is fundamentally intrinsic. This might suggest that u in (8) should be some tangent vector. Furthermore, for a manifold M with dimension greater than three, there is no obvious single vector u (or one dimensional subspace in T_PM) orthogonal to u_1 an u_2, so this initial idea runs into both heuristic and technical difficulties.

[3]Intrinsic curvature is the only kind of curvature which a Riemannian manifold can be expected to have in general, so the adjective "intrinsic" can be left off in this context.

[4]The second fundamental form itself and the extension Q to $T_P\mathcal{S}$ are more properly quadratic forms, but being a quadratic form, as mentioned above, is closely related to a bilinear form B and a linear operator S.

It is difficult (for me at least) to see precisely how the sectional curvatures are (or should be) intertwined or related to one another algebraically. It seems that this difficulty has been resolved by the introduction of a third argument vector, which we can take to be an element $u_3 \in \mathbb{S}_P^{n-1} \subset T_P M$. This vector can be thought of as being used to determine the vector u in (8) in such a way that the following (nonobvious) properties hold:

1. $R_0(u_1, u_2) = Kw$ for some $w \in \text{span}\{u_1, u_2\}$ if $u_3 \in \text{span}\{u_1, u_2\}$, so that there is a function

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which is called the **Riemannian curvature operator**.

   (a) If $u_3 = u_1$, then

$$R(u_1, u_2, u_1) = K(u_1, u_2)\; u_2, \tag{9}$$

   but if $u_3 = u_2$, then

$$R(u_1, u_2, u_2) = -K(u_1, u_2)\; u_1. \tag{10}$$

   Notice that in each case a vector $R(u_1, u_2, u_3)$ is chosen having the form of $R_0(u_1, u_2)$ suggested above, but the vector u involved in that choice depends on the choice of u_3.

   (b) More generally, if $u_3 \notin \text{span}\{u_1, u_2\}$, then $R(u_1, u_2, u_3) \notin \text{span}\{u_1, u_2\}$. Thus, the choice of the third vector determines the disposition of the vector direction of $R(u_1, u_2, u_3)$ in a somewhat predictable way.

2. The Riemannian curvature operator (which Riemann didn't actually know about) is $cC^\infty(M)$ linear in each of the three arguments, so that $R^* : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M)$ by

$$R^*(z_1, z_2, z_3, z_4) = \mu(R(z_1, z_2, z_3), z_4) \tag{11}$$

is a $(0, 4)$ tensor.

These are not quite all the properties required to determine the Riemannian curvature operator R. I will give those properties in the next section. Siffice it to say for now that a relatively short list of properties all but one of which are similar to (9) and (10) do determine a unique operator R and a unique $(0, 4)$ tensor given by (11) called the **Riemannian curvature tensor**. This operator/tensor which captures all the sectional curvatures and is determined by the sectional curvatures is, in some sense, the entire algebraic story of the basic geometric meaning of (intrinsic) curvature for a manifold M.

When M is three dimensional, any two dimensional subspace $\text{span}\{u_1, u_2\}$ of the vector space $T_P M$ is determined by a single unit vector $u \in T_P M$ with u orthogonal to u_1 and u_2. In this case, we define a quadratic form $Q : T_P M \to \mathbb{R}$ by

$$Q(u) = \frac{R^*(u, u_1, u, u_1) + R^*(u, u_2, u, u_2)}{2} = \frac{\mu_P(R(u, u_1, u), u_1) + \mu_P(R(u, u_2, u), u_2)}{2}. \tag{12}$$

Much in the same way the mean curvature of a surface $\mathcal{S} \subset \mathbb{R}^3$ as expressed in (5) is independent of the two orthogonal directions $\mathbf{u}_1 = \dot{\gamma}$ and $\mathbf{u}_2 = \dot{\gamma}^\perp$ in $T_P\mathcal{S}$, in this case it turns out the value of Q is independent of the particular orthonormal basis $\{u_1, u_2\}$ for $(\text{span}\{u\})^\perp$.

Exercise 8 Show the value of the quadratic form $Q : T_P M \to \mathbb{R}$ given in (12) is the trace of a certain linear operator $L : T_P M \to T_P M$. Hint: Use properties (9) and (10) to show $R(u, u, u) = \mathbf{0}$.

Exercise 8 justifies the assertion above that $Q(u)$ is independent of the orthonormal basis $\{u_1, u_2\}$ because the trace of a linear operator (expressed in terms of a particular basis) is independent of the basis.

Continuing with the algebraic and spatial extension of the quadratic form Q above, one obtains a bilinear function $B : T_P M \times T_P M \to \mathbb{R}$ satisfying $B(z,z) = \|z\|^2 Q(z/\|z\|)$ where $\|z\|^2 = \mu_P(z,z)$ as usual. The spatial extension

$$\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M) \qquad \text{by} \qquad \mathrm{Ric}(z_1, z_2) = B(z_1, z_2)$$

to the manifold is $cC^\infty(M)$ (module) linear in each of the two arguments and is thus a $(0,2)$ tensor.[5] Evaluation at a point $P \in M$ can be indicated notationally by

$$\mathrm{Ric}(z_1, z_2)(P) = B_P((z_1)_P, (z_2)_P).$$

Richard Hamilton observed that $\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M)$ is the same kind of tensor as the Riemannian metric tensor $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M)$. This opens the door to the consideration of a one parameter family of metric tensors $g = g(t)$ on a fixed topological manifold X depending differentiably on a real parameter t and satisfying a differential equation

$$\frac{d}{dt} g = \mathrm{Ric}\,.$$

Remember the Riemannian curvature tensor and consequently the Ricci tensor is determined by $g = g(t)$ the Riemannian metric tensor. Such a family corresponds to a one parameter family of Riemannian manifolds $M = M(t) = (X, g(t))$, and ultimately it was the consideration of such deformation families that led to the solution/confirmation of the Poincaré conjecture.

6.4 Higher dimensional Riemannian manifolds

I have promised to give a kind of axiomatic treatment of the algebraic and spatial extension of the (basic geometric) collection of Riemannian sectional curvatures to the Riemannian curvature tensor (which should probably be named after someone else (Hermann Weyl? Christoffel? Levi-Civita? Ricci? Elie Cartan?). Some people definitely feel Riemann understood all about the curvature tensor as understood by these later authors, but it seems clear he never really wrote down the details. Apparently he did make some kind of attempt to do something like writing down the details; see [Dar15].

In any case, I will try to list all the properties that may be associated with the algebraic and spatial extension to obtain a the tensor commonly known as the Riemannian curvuture tensor. Again, it is my perception that the basic geometric object (definitely introduced by Riemann) is the collection of sectional curvatures and the curvature tensor may be viewed as mostly a kind of algebraic organizational device somewhat separate from the geometry. On the other hand, there is an expression for the values of the curvature tensor involving derivatives of (coordinate induced) vector fields, and such derivatives are definitely intrinsically geometric. The formula, however, also has another ingredient whose relation to geometry is much less obvious (at least to me). That additional ingredient is called the "bracket" of vector fields. The bracket certainly has some algebraic properties, but I'm not sure if it should be thought of as entirely algebraic either. One might say the most prominent properties of the bracket are analytic. In any case, the bracket itself deserves a great deal of discussion and contemplation. I do not pursue that here but rather leave it to you.

Finally I will extend the discussion of the Ricci curvuture and the Ricci tensor to higher dimensional manifolds and mention other "named" curvatures, i.e., algebraic summaries of sectional curvature information, including the **scalar curvature** which I have not discussed above.

[5] Different authors use different normalizing constants in the definition of the Ricci tensor. For example, in terms of the notation I have used above, Ben Andrews uses

$$\mathrm{Ric}(z_1, z_2) = 2B(z_1, z_2)$$

for his definition of the Ricci tensor on a three manifold. In this regard, I have followed [dC92].

6.4.1 Curvature tensor

Notice there is an inherent evaluation at a point, say $P \in M$, to calculate the intrinsic curvature $K(u_1, u_2)$ of the submanifold $\mathcal{S}(u_1, u_2)$ in (9) and (10) so that these properties/identities may be taken as statements about (smooth) functions on M. If we wish to emphasize this evalation, we can write R_P and/or K_P to go along with the vectors $(u_1)_P$, $(u_2)_P$ and $(u_3)_P$, but generally in the discussion below I will suppress the evaluation at P giving an implicit spatial extension from the outset.

Remember we have Riemann's definition of the sectional curvature

$$K(u_1, u_2) = \frac{1}{g_{11} \circ \xi} \big[d(\Gamma^2_{11})(u_2) - d(\Gamma^2_{12})(u_1) \\ - \Gamma^1_{12}\, \Gamma^2_{11} - (\Gamma^2_{12})^2 + \Gamma^2_{11}\, \Gamma^2_{22} + \Gamma^1_{11}\, \Gamma^2_{12} \big] \tag{13}$$

given in (6) dependent on two (local) unit fields u_1 and u_2 for which $\{u_1, u_2\}$ is a local orthonormal (moving) frame. The Christoffel symbols here are obtained by

1. taking a totally geodesic submanifold

$$\mathcal{S}(u_1, u_2) = \{P\} \cup \{\gamma(s) : s_0 < s < s_0 + \epsilon,\ [\dot\gamma(s_0)] \in \text{span}\{u_1, u_2\} \cap \mathbb{S}^{n-1}_P\}$$

   where γ parameterizes a unit speed geodesic with $\gamma(s_0) = P$ and ϵ is some positive number, and

2. restricting the Riemannian metric tensor μ_P to the two dimensional submanifold $\mathcal{S}(u_1, u_2)$.

The full list of properties of the Riemannian curvature tensor, which may be taken as axioms for the sake of algebraic extension/organization of the sectional curvatures, are the following:

RCT1 $R(u_1, u_2, u_1) = K(u_1, u_2)\ u_2$ for every orthonormal frame $\{u_1, u_2\}$. Remember, the "u_1" in the third argument of R here tells you to organize the sectional curvature as the scalar in front of the "other" basis vector u_2. If a different vector appears in the third argument, the tensor will tell you some different thing to do with $K(u_1, u_2)$.

RCT2 (tensorial property) $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(M)$ is $cC^\infty(M)$ (mudule) linear in each of the three arguments.

RCT3 $R(u_2, u_1, u_3) = -R(u_1, u_2, u_3)$ for all orthonormal frames $\{u_1, u_2, u_3\}$.

RCT4 $\mu(R(u_1, u_2, u_4), u_3) = -\mu(R(u_1, u_2, u_3), u_4)$ for all orthonormal frames $\{u_1, u_2, u_3, u_4\}$.

RCT5 $\mu(R(u_3, u_4, u_1), u_2) = \mu(R(u_1, u_2, u_3), u_4)$ for all orthonormal frames $\{u_1, u_2, u_3, u_4\}$.

RCT6 (Bianchi identity) $R(u_1, u_2, u_3) + R(u_2, u_3, u_1) + R(u_3, u_1, u_2) = \mathbf{0}$ for all orthonormal frames $\{u_1, u_2, u_3\}$.

Properties **RCT3-RCT5** are "symmetry" properties, though **RCT6** is often included as a symmetry property as well. Each of the properties merits careful consideration, though they are presented here as axioms. If they are considered in the context of the derivative/bracket formula given below the possibility of geometric justification makes more sense. Notice that properties **RCT4** and **RCT5** as stated here only make sense for manifolds of dimension $n = 4$ and higher.[6]

[6]In view of the tensorial property, the symmetry properties can also be stated in terms of arbitrary collections $\{z_1, z_2, z_3\}$ and/or $\{z_1, z_2, z_3, z_4\}$ in $\mathfrak{X}(M)$. In this way, one can obtain some notion of a degenerate Riemannian curvature tensor in the lower dimensions $n = 3$ and $n = 2$ or even $n = 1$. Thus, some authors, Ben Andrews for example, go so far as to contradict what I have written above asserting that a one dimensional Riemannian manifold has "zero curvature." I will leave it to you to determine which perception you wish to embrace. Certainly Ben Andrews should be taken seriously.

The basic extension/existence theorem from this point is that properties **RCT1-RCT6** determine a unique operator $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and a unique $(0, 4)$ tensor

$$R^* : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X} \to cC^\infty(M)$$

given by

$$R^*(z_1, z_2, z_3, z_4) = \mu(R(z_1, z_2, z_3), z_4).$$

In a certain sense that is the end of the story. At least that is a definition.

6.4.2 Formulas, commutators, and the bracket

I will mention two more things. The first is that the Riemannian curvature operator can be expressed in terms of intrinsic derivatives of vector fields, that is, there is a formula which can be derived from the axiomatic properties **RCT1-RCT6** given above.

Perhaps the simplest (or at least quickest) way to describe the relation is locally in terms of chart induced fields. As usual, let $v_j = d\mathbf{p}(\mathbf{e}_j)$ for $j = 1, 2, \dots, n$ be the fields induced by a chart/chart function pair $(U, \mathbf{p})$ at P. Then $dv_k : \mathfrak{X}(\mathbf{p}(U)) \to \mathfrak{X}(\mathbf{p}(U))$ and

$$R(v_i, v_j, v_k) = d[dv_k(v_j)](v_i) - d[dv_k(v_i)](v_j). \tag{14}$$

The vector field on the right of (14) should not be confused with

$$dv_k[dv_j(v_i) - dv_i(v_j)] \tag{15}$$

which is the zero vector field due to the symmetry of the Riemannian connection.

Something more can be said. If we take the expression on the right of (14) and adapt it to vectors u_i, u_j and u_k from an orthonormal frame $\{u_1, u_2, \dots, u_n\}$, then we get a notationally prettier form

$$\nabla_{u_i}\nabla_{u_j}u_k - \nabla_{u_j}\nabla_{u_i}u_k = (\nabla_{u_i}\nabla_{u_j} - \nabla_{u_j}\nabla_{u_i})u_k \tag{16}$$

though this expression does not constitute a formula for the curvature operator. If we adapt similarly the expression (15) we see

$$du_k(\nabla_{u_i}u_j - \nabla_{u_j}u_i) \tag{17}$$

and get some (other) nonzero vector field. It turns out that Riemannian curvature, or at least the algebraic organization of Riemannian curvature, has fundamentally to do with the difference between these two vector fields.

Focusing on the inner expression

$$\nabla_{u_i}u_j - \nabla_{u_j}u_i$$

in (17) we see the kind of vector field expression required (by the symmetry of the Riemannian connection) to vanish on chart induced vector fields. This secondary commutator field turns out to be an interesting field in general. It is usually denoted by

$$[z_1, z_2] = dz_2(z_1) - dz_1(z_2) \tag{18}$$

for any $z_1, z_2 \in \mathfrak{X}(M)$ and is called the **bracket** of the vector fields z_1 and z_2.

With the bracket in hand, we can give a formula for the curvature operator (more or less) in terms of directional derivatives: For orthonormal vector fields u_1, u_2, u_3 the formula (14) takes the form

$$R(u_1, u_2, u_3) = \nabla_{u_1}\nabla_{u_2}u_3 - \nabla_{u_2}\nabla_{u_1}u_3 - du_3(\ [u_1, u_2]\ ). \tag{19}$$

Thus (16) gives one vector field and (17) gives another vector field, and R is seen to measure the difference of these two vector fields. In words the Riemannian curvature operator on a vector field w gives the difference between the commutator field

$$\nabla_{u_1}\nabla_{u_2}w - \nabla_{u_2}\nabla_{u_1}w$$

and the differential of w on the commutator/bracket

$$dw(\ [u_1, u_2]\ ).$$

Several features of this formula may be noted at this point.

Given any $w \in \mathfrak{X}(M)$ the commutator expression

$$\nabla_{u_1}\nabla_{u_2}w - \nabla_{u_2}\nabla_{u_1}w = (\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1})w$$

gives a vector field. The commutator expression $\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1}$ is a well-defined element of the module $\mathcal{L}(\mathfrak{X}(M) \to \mathfrak{X}(M))$ with $\mathfrak{X}(M)$ considered as a linear space over $\mathbb{R}$. With all these derivatives involved we should not expect $\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1}$ to be module linear, and it is not. One should not expect the commutator $\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1}$ to be Leibnizian either. Specifically, if one focuses on the (module) subspace $\mathcal{LL}(\mathfrak{X}(M) \to \mathfrak{X}(M))$ of Leibnizian operators in $\mathcal{L}(\mathfrak{X}(M) \to \mathfrak{X}(M))$, then the operators $\nabla_{u_1}\nabla_{u_2}$, and $\nabla_{u_2}\nabla_{u_1}$ are not typically in there, and there is no real reason to expect the commutator $\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1}$ is in there either. In particular, there is no real expectation that there exists a (local) unit vector field u for which

$$(\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1})w = \nabla_u w \qquad \text{for} \qquad w \in \mathfrak{X}(M)$$

or even any vector field $z \in \mathfrak{X}(M)$ for which

$$(\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1})w = dw(z) \qquad \text{for} \qquad w \in \mathfrak{X}(M)$$

On the other hand,

$$dw(\ [u_1, u_2]\ ) = \|[u_1, u_2]\| dw\left(\frac{[u_1, u_2]}{\|[u_1, u_2]\|}\right) = \nabla_{[u_1,u_2]/\|[u_1,u_2]\|}w$$

is Leibnizian in the argument w, and $R(u_1, u_2, w)$ is (module) linear in the third argument. This means

$$(\nabla_{u_1}\nabla_{u_2} - \nabla_{u_2}\nabla_{u_1})w = R(u_1, u_2, w) - dw(\ [u_1, u_2]\ )$$

is the difference of a (module) linear operator and a Leibnizian operator. This suggests perhaps that there is some interesting structure (at least some algebraic structure) involving these operators inside $\mathcal{L}(\mathfrak{X}(M) \to \mathfrak{X}(M))$.

Note the expression $\nabla_{u_i}\nabla_{u_j}w$ for $w \in \mathfrak{X}(M)$ corresponds to the second derivative expression

$$\frac{\partial^2 \mathbf{w}}{\partial x_i \partial x_j}$$

for a traditional vector field $\mathbf{w}$ on flat space $\mathbb{R}^n$, and there would be absolutely no reason to believe this vector field is expressible as $d\mathbf{w}(\mathbf{z})$ for any vector field $\mathbf{z}$ on $\mathbb{R}^n$. On the other hand,

$$\frac{\partial^2 \mathbf{w}}{\partial x_i \partial x_j} - \frac{\partial^2 \mathbf{w}}{\partial x_j \partial x_i} \equiv \mathbf{0},$$

so while the corresponding traditional commutator is easily expressible as the differential of $\mathbf{w}$ on a vector field $\mathbf{z} = \mathbf{0}$, this is definitely not a quantity which draws any particular interest or provides any obvious motivation for generalization. This means it is going to be difficult to get any idea of what interesting thing is going on with these fields by looking at flat space.[7]

On a general manifold however both vector fields

$$d[dz_3(z_2)](z_1) - d[dz_3(z_1)](z_2) \tag{20}$$

and

$$[z_1, z_2] = dz_2(z_1) - dz_1(z_2) \tag{21}$$

are typically nonzero and of interest. Obviously the bracket has some intrinsic geometric meaning as

$$[z_1, z_2] = \|z_1\| \; \nabla_{z_1/\|z_1\|} z_2 - \|z_2\| \; \nabla_{z_2/\|z_2\|} z_1,$$

but with this level of "algebraic entwinement" of the directional derivatives it becomes somewhat difficult to see the bracket as fundamentally geometric. This is probably the reason the "bracket" has a name that is so lacking in informative content. Perhaps the "commutator" would be a better name for the quantity in (21). In any case,

$$[\,\cdot\,,\;\cdot\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is a well-defined function. This function has certain algebraic and analytic properties which deserve close attention, but I will reserve that discussion for another place and time (and document)—or even better, I will leave careful consideration of the bracket operation to you.

The formula giving the Riemannian curvature operator on general vector fields $z_1, z_2, z_3 \in \mathfrak{X}(M)$ is

$$R(z_1, z_2, z_3) = d[dz_3(z_2)](z_1) - d[dz_3(z_1)](z_2) - dz_3(\;[z_1, z_2]\;). \tag{22}$$

Notice this is the difference of the "commutator" vector field given in (20) and a differential in the direction of the bracket/commutator vector field $[z_1, z_2]$ given in (21) resulting in an expression

$$d[dv_3(v_2)](v_1) - d[dv_3(v_1)](v_2) - d_{[v_1,v_2]}v_3 = d[dv_3(v_2)](v_1) - d[dv_3(v_1)](v_2)$$

for chart indced vector fields v_1, v_2, v_3 and satisfying

$$d[dv_3(v_2)](v_1) - d[dv_3(v_1)](v_2) - d_{[v_1,v_2]}v_3 = \mathbf{0} \tag{23}$$

when $M = \mathbb{R}^n$ is flat Euclidean space.

Exercise 9 Show the formula (22) vanishes for

$$\begin{aligned} z_1 &= v_i = d\mathbf{p}(\mathbf{e}_i) \\ z_2 &= v_j = d\mathbf{p}(\mathbf{e}_j) \\ z_3 &= v_k = d\mathbf{p}(\mathbf{e}_k) \end{aligned}$$

$i, j, k = 1, 2 \ldots, n$ whenever $\mathbf{p} : U \to \mathbb{R}^n$ is a chart function for flat Euclidean space $M = \mathbb{R}^n$. Thus, the Riemannian curvature operator is measuring the deviation of a manifold from flat space $M = \mathbb{R}^n$. (At least if R is non-vanishing at a point $P \in M$, then you know M is not isometric to $\mathbb{R}^n$.)

[7]This is not surprising since flat space is flat, and one shouldn't expect to see anything interesting having to do with curvature in that context.

6.4.3 Ricci tensor and scalar curvature

Though I have compared the Ricci curvature, given at a point $P \in M$ in a three manifold $M = M^3$ by the sum of the sectional curvatures of two (totally geodesic) submanifolds, to the mean curvature of a surface $\mathcal{S} \subset \mathbb{R}^3$ given by the average of orthogonal normal curvatures, the Ricci curvature is an inherently much more complicated quantity. First of all, the orthogonal geodesic filaments $[\gamma_1]$ and $[\gamma_2]$ used to determine the mean curvature lie in the same two dimensional submanifold $\mathcal{S}$. The two geodesic surfaces $\mathcal{S}(u, u_1)$ and $\mathcal{S}(u, u_2)$, on the other hand, used to determine the Ricci curvature are distinct two dimensional submanifolds of $M = M^3$. Most importantly, $\mathcal{S} \subset \mathbb{R}^3$ stands alone as a monolith while there is an entire sphere's worth of two manifolds corresponding to $u \in \mathbb{S}^2_P \subset T_PM$ leading to distinct averages, and distinct values for $\text{Ric}(u, u)$ at $P \in M = M^3$. For a schematic representation of this comparison/distinction see Figure 3.

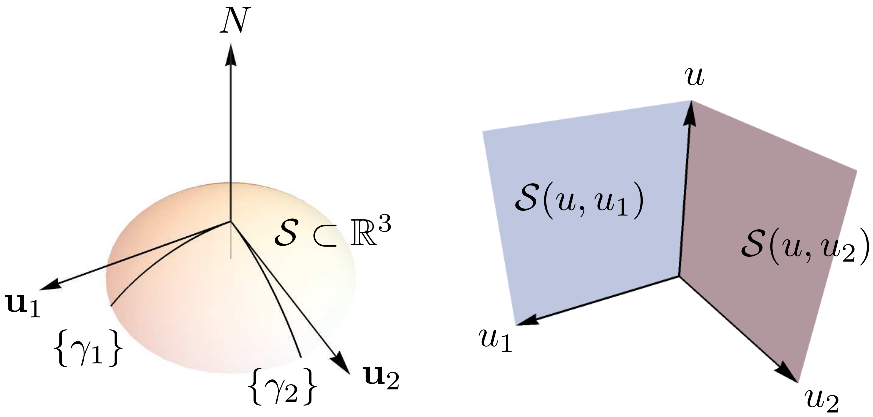

Figure 3: Orthogonal directions for normal curvatures of a surface $\mathcal{S} \subset \mathbb{R}^3$ (left); the average of these values is the mean curvature $H = (k_{\gamma_1} + k_{\gamma_2})/2$ and is independent of the orthogonal filaments. Schematic diagram for orthogonal directions determining sectional curvatures and the Ricci curvature of a three dimensional manifold $M = M^3$ (right); in this illustration the manifold M is represented schematically by the ambient $\mathbb{R}^3$ and the two dimensional submanifolds $\mathcal{S}(u, u_1)$ and $\mathcal{S}(u, u_2)$ determined by the directions u_1 and u_2 are represented schematically by planes; the average $(K(u, u_1) + K(u, u_2))/2$ of the sectional curvatures in independent of the orthonormal filaments $u_1 = [\gamma_1]$ and $u_2 = [\gamma_2]$.

Note carefully that in the correspondence I've suggested, the (intrinsic) vector u corresponds to the extrinsic normal N while the submanifolds $\mathcal{S}(u, u_1)$ $\mathcal{S}(u, u_2)$ correspond to the geodesics paraemterized by γ_1 and γ_2. Finally, the (intrinsic) sectional curvatures $K(u, u_1)$ and $K(u, u_2)$ correspond to the extrinsic normal curvatures k_{γ_1} and k_{γ_2}. While there is essentially only a circle of extrinsic unit vectors $\{bu_1, \mathbf{u}_2\} \subset \mathbb{S}^1_P$, and corresponding normal curvatures k_{γ_1} and k_{γ_2}, to consider for a surface $\mathcal{S} \subset \mathbb{R}^3$, there is an $n-1$ (at least $n - 1 = 2$) sphere of starting vectors u for $\text{Ric}(u, u)$, and for each such u, there is at least an $n - 2$ sphere of unit vectors $\{u_1, u_2\}$, and corresponding to sectional curvatures $K(u, u_1)$ and $K(u, u_2)$ to consider at each point on an n dimensional Riemannian manifold M.

For an n dimensional manifold $M = M^n$ (12) becomes

$$Q(u) = \frac{1}{n-1} \sum_{j=1}^{n-1} R^*(u, u_j, u, u_j) = \frac{1}{n-1} \sum_{j=1}^{n-1} \mu_P(R(u, u_j, u), u_j) = \frac{1}{n-1} \sum_{j=1}^{n-1} K(u, u_j) \tag{24}$$

where $\{u_1, u_2, \ldots, u_{n-1}, u\}$ is an orthonormal basis for T_PM. The function Q is well-defined and extends to a quadratic form on T_PM. The quadratic form leads to a bilinear form $B : T_PM \times T_PM \to \mathbb{R}$ and eventually to a two form field

$$\text{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to cC^\infty(\mathbb{R}) \qquad \text{by} \qquad \text{Ric}(z_1, z_2) = B(z_1, z_2).$$

The number $\mathrm{Ric}(u,u) = Q(u)$ at a point P is often referred to as the **Ricci curvature** of M at P "in the direction" u.

Another algebraic combination of sectional curvatures at a point is given by what is called the **scalar curvature**

$$K_{\mathrm{avg}} = \frac{1}{n}\sum_{i=1} \mathrm{Ric}(u_i,u_i) = \frac{1}{n(n-1)}\sum_{i,j=1}^{n} \mu(R(u_i,u_j,u_i),u_j)$$

which gives, in a certain sense, the average of all sectional curvatures at a point $P \in M$.

I guess I will leave it there.

References

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[dC92] Manfredo do Carmo. *Riemannian Geometry*. Birkhäuser, Boston, MA, first edition, 1992.