1 Show that a local homeomorphism \( \phi : X \to Y \) induces unique lifting of paths, i.e., if \( \gamma : [a, b] \to Y \) is a path connecting points \( \phi(x) \) to \( \phi(y) \), then there is a unique path \( \hat{\gamma} : [a, b] \to X \) with \( \hat{\gamma}(a) = x \) and
\[
\phi \circ \hat{\gamma} = \gamma.
\]
Can one conclude that \( \hat{\gamma}(b) = y \)?

**Oops!** This is not true as shown by ???’s counterexample:

Let \( \phi : \mathbb{R}^2 \to \mathcal{C} \) where \( \mathcal{C} = \{ (\cos \theta, \sin \theta, z) : \theta, z \in \mathbb{R} \} \) is a cylinder in \( \mathbb{R}^3 \) and
\[
\phi(x, y) = (\cos \theta(x), \sin \theta(x), y)
\]
where \( \theta(x) = \pi x + \sqrt{\pi^2 x^2 + 2\pi} \). Note that
\[
x = \frac{\theta}{2\pi} - \frac{1}{\theta}
\]
so that \( \theta \searrow 0 \) as \( x \searrow -\infty \) and \( \theta \approx x/(2\pi) \) when \( x \to \infty \). Thus, the path \( \gamma : [0, 1] \to \mathcal{C} \) by \( \gamma(t) = (\cos(\pi(1-2t)/2), \sin(\pi(1-2t)/2), 0) \) with lift starting at \( \phi(2/\pi - 1/4, 0) = (0,1,0) \) must lift on \([0, 1/2]\) to
\[
\hat{\gamma}(t) = \left( \frac{\pi(1-2t)^2 - 8}{4\pi(1-2t)}, 0 \right).
\]
It follows that \( \hat{\gamma}(t) \to (\infty, 0) \) as \( t \searrow 1/2 \). Therefore \( \hat{\gamma} \) cannot be a globally defined (continuous) lifting.

Let’s modify this problem as follows:
Definition 1 A surjective continuous map $\phi : X \rightarrow Y$ of topological spaces is said to be a covering map and $X$ is called a covering space of $Y$ if for each $y \in Y$ there is a neighborhood $V$ of $y$ such that

$$\phi^{-1}(V) = \bigcup_{\alpha} U_{\alpha}$$

for some disjoint open sets $U_{\alpha} \subset X$ and $\phi \big|_{U_{\alpha}}$ is a homeomorphism onto $V$ for each $\alpha$.

(a) Show that a covering map is a local homeomorphism which admits unique lifting of paths.

(b) Show that the Gauss map on a compact surface in $\mathbb{R}^3$ with positive Gauss curvature is a covering map onto $S^2$.

The counterexample above gives a situation in which there is, in fact, no lifting of a path. Can you give an example of a local homeomorphism $\phi : X \rightarrow Y$ which admits path lifting but provides another counterexample to the original problem due to loss of uniqueness? For further discussion of covering maps and arc lifting maps see M.P. do Carmo’s book *Differential Geometry of Curves and Surfaces*, pp. 371–384.

2 Given two points $x$ and $y$ in $\mathbb{R}^n$, interpret geometrically the condition $x = t(x - y)$ for some $t \in \mathbb{R}$.

3 Prove that a bijective continuous function $f : X \rightarrow Y$ of a compact topological space $X$ onto a Hausdorff topological space $Y$ is a homeomorphism.

4 Explain the ring structure of the smooth functions on a manifold.

5 Let $T$ denote the collection of all topological charts for an $n$-dimensional manifold $M$ and let $\mathcal{A}$ denote the maximal atlas of charts making $M$ a differentiable manifold. Given two distinct points in $p_1, p_2 \in M$ and any real numbers $c_1, c_2$, prove there is a smooth function $f : M \rightarrow \mathbb{R}$ such that $f(p_j) = c_j$. (This means there a lot of smooth functions in $\mathcal{F}(M)$.) Prove there is no function $f : M \rightarrow \mathbb{R}$ for which $f \circ \xi^{-1}$ is smooth for every $\xi \in T$. 

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